members living in the eastern part of Pennsylvania, southern New Jersey, and Delaware, the meetings usually to take place in Philadelphia. Professor H. H. MITCHELL was elected chairman for the coming year, and Professor A. A. BENNETT vice-chairman and secretary-treasurer. It was voted to defer official adoption of by-laws until the next meeting, which is scheduled for November 26, 1927. After a vote of appreciation for the courtesies extended by Lehigh University, the session adjourned.

The following papers were presented:

1. "The evolutes of a certain type of symmetrical plane curves," by Professor J. B. REYNOLDS, Lehigh University.

2. "The analogue for ideals of the Lagrange-Gauss theory of quadratic forms," by Professor H. H. MITCHELL, University of Pennsylvania.

3. "A new treatment of exponentials and logarithms on the basis of a modified Dedekind theory of irrationals," by Professor L. L. SMAIL, Lehigh University.

4. "The derivation and solution of certain ordinary differential equations," by Professor W. M. SMITH, Lafayette College.

5. "The state course of study in mathematics," by J. A. FOBERG, State director of mathematics and science.

Abstracts of these papers are given below, the numbers corresponding to those in the list of titles.

1. The author demonstrated nine theorems concerning a curve and its evolutes when the curve is an analytic symmetrical plane curve having continuous evolutes, one infinite branch and no point singularities. The parabola and its successive evolutes are a case of the type of curves discussed.

2. Professor Mitchell discussed the determination of the number of classes of ideals in the quadratic field by methods that follow closely those used in the theory of binary quadratic forms. He pointed out that in this case these methods lead to more satisfactory results than those based on the Minkowski theorems concerning linear forms.

3. In the published literature on the theory of irrational numbers on either the Dedekind or the Cantor definition, the four rational operations are discussed by most writers on the basis of the original definition of irrational numbers. Before taking up the higher operations of exponentials and logarithms, however, they all introduce the theory of limits with more or less of its definitions, terminology and rules of manipulation, and then base the treatment of the rational and the irrational exponents and logarithms on this foundation. From several points of view it would seem desirable to avoid this break in the discussion of the rational and irrational operations caused by interpolating the theory of limits at this point. The object of the paper by Professor Smail is to show how, by use of a modified form of the Dedekind definition of irrationals,

it is possible to treat the rational and irrational exponents and logarithms without explicit use of the theory of limits, using only the original definition of irrational numbers.

4. This expository paper discussed the derivation and solution of the Riccati equation; of the equation of forced, damped vibrations; and of a certain linear equation with discontinuous solutions arising from a kind of maintained vibration in which the force of restitution or "spring" of the body, susceptible of vibration, is subject to an imposed periodic variation.

5. The attention consistently given by the Mathematical Association to the interests of mathematical instruction in the secondary school makes it appropriate that discussion of mathematics instruction in the public schools should form part of the program of this initial meeting of the Philadelphia Section. The state program of studies in Pennsylvania contemplates a continuous twelve-year program in such major subjects as mathematics, science, English, social studies, art. Mathematics is a required study through the first nine school years—thereafter it is elective. Opportunity is afforded in the junior high school for presenting such topics in the whole field of elementary mathematics as can be justified on the basis of their values in training for citizenship. Obviously, the design of the course of study involves much experiment and study on the part of teacher and supervisor. Such special objectives as meeting college requirements are taken care of in the senior high school. A number of colleges and universities in Pennsylvania now admit applicants upon a showing of twelve units of work done in the three-year senior high school. It is hoped and expected that this plan will become general in the near future.

A. A. BENNETT, *Secretary*

THE FIFTEENTH MEETING OF THE IOWA SECTION

The fifteenth regular meeting of the Iowa Section of the Mathematical Association of America was held at Coe College, Cedar Rapids, Iowa, on April 30 and May 1, 1926, in affiliation with the annual meeting of the Iowa Academy of Science.

Among those attending were the following twenty-nine members of the Association: E. W. Chittenden, L. M. Coffin, Julia T. Colpitts, I. S. Condit, Marian E. Daniells, R. D. Daugherty, C. W. Emmons, Annie W. Fleming, C. Gouwens, D. L. Holl, Dora E. Kearney, F. M. McGaw, J. V. McKelvey, C. A. Messick, E. E. Moots, E. A. Pattengill, J. F. Reilly, F. Reusser, H. L. Rietz, E. R. Smith, G. W. Snedecor, J. Theobald, J. S. Turner, L. E. Ward, C. W. Wester, W. H. Wilson, R. Woods, C. C. Wylie, J. L. Yothers.

The chairman of the section, Professor I. S. Condit, presided at both the Friday afternoon and the Saturday morning sessions. Dinner was enjoyed together Friday evening at the Y. M. C. A. building, at which time Professor Condit gave his retiring address on the subject "Progress of the Perry Movement". At the business meeting the following were elected officers for 1926–1927: Chairman, J. V. McKelvey, Iowa State College; Secretary-treasurer, J. F. Reilly, University of Iowa.

The next regular meeting will be held at Iowa City, May 6, 7, 1927.

The following papers were presented:

1. "Expansions of a function in the neighborhood of branch points" by Professor C. Gouwens, Iowa State College.
2. "A preliminary report on freshman mathematics for social science students" by Professor C. W. Emmons, Simpson College.
3. "A general interpolation formula" by Professor J. F. Reilly, University of Iowa.
4. "On a certain group of two-squares" (by title) by Professor W. J. Rusk, Grinnell College.
5. "On certain three-squares that are also two-squares" (by title) by Professor Rusk.
6. "Note on the concepts of correlation and of stochastic connection" by Professor H. L. Rietz, University of Iowa.
7. "Some relativistic reflections" (by invitation) by Professor E. S. Allen, Iowa State College.
8. "Progress of the Perry movement" by Professor I. S. Condit, Iowa State Teachers College.
9. "The torsion problem and its analog in curvilinear coordinates" by Dr. D. L. Holl, Iowa State College.
10. "Note on problem solving in elementary mathematics" by Professor C. W. Wester, Iowa State Teachers College.
11. "Note on ciphering matches" by Professor Wester.
12. "Fundamental relations for $\sin z$ and $\cos z$ defined by infinite products" by Professor E. R. Smith, Iowa State College.
13. "Certain objectives in the teaching of calculus" by Professor W. H. Wilson, University of Iowa.
14. "Solutions of a certain functional equation" by Professor Wilson.
15. "The sunspot cycle in the flow of rivers" by Professor C. C. Wylie, University of Iowa.
16. "The solar surface during the past five years" (by title) by Mr. D. E. Hadden, Alta, Iowa.
17. "Geometric interpretation of median and quartiles" (by title) by Professor E. E. Watson, Iowa State Teachers College.

18. "A slide rule for statisticians" by Professor G. W. SNEDECOR, Iowa State College.

19. "An anabasis to signed quantities" by Professor R. D. DAUGHERTY, Iowa State Teachers College.

Abstracts of these papers are given below, the numbers corresponding to those in the list of titles.

1. In his paper Professor Gouwens gave an exposition and discussion of the Newton polygon method for the expansion of an algebraic function.

2. The report gives the purpose and the plan of a new course in mathematics which Professor Emmons proposes to offer at Simpson College. The topics included are selected from the traditional courses in college mathematics with a view to their utility for students of the social sciences.

3. In his paper Professor Reilly developed a third order difference interpolation formula containing two parameters. By assigning special values to the parameters a number of formulas already known were obtained, in particular Sprague's osculatory formula. A method for determining the parameters so as to produce the best interpolated value at any point in a limited range was given by the use of least squares.

4. In his first paper Professor Rusk, assuming that the square of each of two numbers can be expressed as a three square, obtains in a very elementary fashion a set of forty-eight functions of their elements that are two squares. From these he is able to give explicit form to the product of any two numbers as a four-square.

5. In his second paper Professor Rusk assumes that the two numbers are the same and is able to get a set of three-squares that are two-squares, in particular the theorem that if $u^2=x^2+y^2+z^2$, then $(x\pm y)^2+(y\pm z)^2+(z\pm x)^2$ is a two-square for any combination of the signs.

6. The paper by Professor Rietz discusses Tschuprow's conception of stochastic connection[1] in comparison with various definitions of correlation given in the literature. It is shown that certain definitions make the concepts of correlation and of stochastic connection equivalent, but that with some of the usual definitions, as pointed out by Tschuprow, there may exist stochastic connection when there is no correlation.

7. Several authors have seen in the doctrine of relativity evidence for philosophical theories, in particular for idealism and for freedom in nature. In the opinion of Professor Allen these interpretations of the implications of relativity seem unjustified; at most, there must be a revision of the definition of causation and of the formulation of idealism.

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[1] For a definition of stochastic connection, see A. A. Tschuprow's *Grundbegriffe und Grundprobleme der Korrelationstheorie* (1925); p. 20.

8. In his address Professor Condit discussed the progress of the Perry movement first as to the gains that have been made during the last quarter century, and second as to objectives. These may be enumerated as follows:

Gains of the quarter century 1900–1925 (a) Motivation of secondary mathematics, (b) Development of scientific spirit in determining standards and measuring results of teaching, (c) Efforts to unify mathematical subjects, and mathematics and science, (d) Development of general mathematics for the junior high school, (e) Efforts to coordinate secondary and college mathematics, (f) Efforts to determine purpose and essential subject matter of college mathematics, (g) More harmonious relations between secondary and college teachers, (h) Determining standards of teacher preparation, (i) Rapid growth of associations of teachers of mathematics and science.

Objectives (a) A greater willingness on the part of teachers to investigate and experiment with all constructive criticism of present practice in teaching mathematics, (b) Preparation of teachers of mathematics who can teach general mathematics efficiently, or who can apply the results of scientific investigation to stratified courses, (c) Adequate training of teachers of mathematics, (d) Better adjustment of college entrance requirements, (e) Determination of essential subject matter in freshman mathematics.

9. The conjugate cylindrical coordinates characterized by $x+iy=F(\alpha+i\beta)$ offer an extension to the two-dimensional solution for the torsion problem of a cylindrical rod. Dr. Holl considers a more general case with circular cross sections, but with meridian curves $r=f(z)$, where the coordinate system $z+ir=F(\alpha+i\beta)$ is adaptable. In the former case the deflection and slope of a soap film suspended from a plane boundary whose contour is that of a cross section of the rod, offer an experimental determination of the direction and magnitude of the stress in the rod.

10. In his first paper Professor Wester gave a preliminary report of an experiment in the use of the equation in the lower grades.

11. The purpose of mathematical contests was stated by Professor Wester to be (a) to arouse interest in mathematics, (b) to furnish an added incentive for the mastery of the more difficult processes, (c) to discover the minds capable of developing a high efficiency and to give them a training that will develop them to their utmost, (d) to arouse school spirit and to create a tradition of achievement in mathematical work.

Attention also was called to the need for standard rules for conducting such contests and for judging results, to the desirability of stage managing them to make them interesting and even exciting to onlookers, and to the need on the part of coaches to study and teach the most efficient computation methods.

12. Professor Smith showed that trigonometry for complex values of z may be logically developed directly from the definition of $\sin z$ in terms of an infinite

product without the aid of series or other infinite expressions. In particular the addition formulas may be developed, and the usual relations for the functions found in trigonometry or calculus may be obtained.

15. By comparing the figures for the run-off of various rivers with the Wolf sunspot numbers, and by forming the mean curves for one cycle, smoothed by overlapping means, Professor Wylie finds that, in general, for rivers west of the Mississippi there is a positive correlation, that is, the most water flows down the river in years when sun spots are numerous. For those investigated in the East a negative coefficient was obtained. The highest positive coefficient was obtained in Iowa, the coefficient for the Iowa river being $+0.97$; and the highest negative coefficients were obtained for New England rivers, Lake Cochituate yielding -0.93, and the Merrimac river -0.90.

16. In his paper Mr. Hadden gave a review of sunspot observations made at Alta, Iowa, from 1921 to 1925.

18. In describing the slide rule that he had invented Professor Snedecor pointed out that it facilitated the calculation of many of the functions employed in statistics. In addition to the usual scales, it contains the Kelley scales of the logarithms of the reciprocals of $(1-r)^{\frac{1}{2}}$ and $(1-r^2)$, together with a scale beginning with log .6745.

19. The introduction of the complex plane near the beginning of algebra was advocated by Professor Daugherty. By the operations of extension and rotation the processes that signed quantities are usually subjected to could be performed.

J. F. REILLY, *Secretary*

TENTH ANNUAL MEETING OF THE KENTUCKY SECTION

The tenth annual meeting of the Kentucky Section of the Association was held May 8, 1926 at Berea College, Berea, Kentucky. Professor T. A. Martin presided at both sessions. A twelve o'clock dinner was served by Berea College at Boone Tavern. At this dinner Dr. W. J. Hutchins, president of Berea College, welcomed the members and their friends to Berea. He related entertaining stories and told interesting facts about the college.

At the business meeting which followed the program, a committee was authorized to consider the advisability of affiliating with the Kentucky Academy of Science; and the section expressed its appreciation to Berea College for the hospitable reception and delightful dinner given to the attending members. Dean P. P. BOYD of the University of Kentucky was made chairman and Professor A. R. FEHN of Centre College was made secretary-treasurer for the year 1926–27. It was voted to have the next meeting at the University of Ken-

tucky. There were twenty-seven present at the meeting, including the following members of the Association: M. C. Brown, J. M. Davis, H. H. Downing, A. R. Fehn, J. M. Guilliams, M. R. Hutcherson, Flora E. LeStourgeon, T. A. Martin, J. N. Peck, R. I. Pepper, E. L. Rees, D. C. South, D. O. Streyffeler.

Papers were presented as follows:

1. "Some remarks on functional calculus" by Dr. FLORA E. LESTOURGEON, University of Kentucky.

2. "Line complexes associated with the motion of a rigid body" by Professor E. L. REES, University of Kentucky.

3. "Unique and non-unique factorization" by Professor W. R. HUTCHERSON, Berea College.

4. "Kreisverwandtschaft" by H. W. MOBLEY, Graduate student, University of Kentucky (by invitation).

5. "Groups of linear transformations in the ternary field" by Miss VEDA L. NELSON, Graduate student, University of Kentucky (by invitation).

6. "Improving scholarship in mathematics in secondary schools" by Principal F. A. SCOTT, Paris High School, Paris, Ky.

7. "Some points on teaching secondary mathematics" by Professor J. M. GUILLIAMS, Berea Normal School.

8. Discussion of the last two papers.

Abstracts of most of these papers are given below, the numbers corresponding to those in the list of titles.

1. This paper discusses functionals having continuity of order one, and derives for such functionals a first and second necessary condition for a minimum analogous to the Euler and Jacobi conditions of the calculus of variations.

2. This was an outline of the kinematical theory of line complexes. It was shown by simple geometric reasoning that orthogonal conjugate lines with respect to the linear complex consisting of lines normal to the trajectories of their points belong to the tetrahedral complex of tangents and that they are the vertex tangents of focal parabolas enveloped by lines of the tetrahedral complex and the characteristics of the planes of these parabolas.

3. Definitions of field, algebraic integer, unit of field were given. In field $K(-3)$ it was shown that factorization was unique, using the factoring of 21 as an example:

$$21=(\sqrt{-3})\ (-\sqrt{-3})\ (2+1\sqrt{-3})\ (2-1\sqrt{-3}),$$

where all of the four expressions are prime and factors of 21. Then in field $K(-5)$ it was shown that factorization was non-unique. Thus,

$$21=3.7=(1+2\sqrt{-5})\ \ (1-2\sqrt{-5}).$$

In this field both 3 and 7 are prime, as well as the other factors.

6. The purpose of the writer was to note criticisms of the teaching of mathematics in the secondary schools and then to offer suggestions which

might remedy the situation. (1) Teachers are not properly prepared. (2) There is a lack of thoroughness in teaching. Teachers should not be in too much of a hurry to cover a certain portion of the book. Suggestions for vitalizing their teaching can be had by reference to the report of the National Committee on the Reorganization of Mathematics in Secondary Schools. (3) Pupils are passed without knowing the subject. Not true in general, but, where true, a rearrangement of courses would correct the evil. Two years is not sufficient to lay a good foundation for collegiate mathematics. (4) New courses and extra-curricular activities crowd out the subject. This should appeal as a challenge to the teachers; they must really teach. Extra-curricular activities offer one of the best opportunities to the teacher to promote interest in mathematics, through the use of mathematics clubs. Finally every teacher should join the Kentucky Section, should exchange ideas with other teachers, should subscribe either to the *Mathematics Teacher* or to *School Science and Mathematics*.

7. An outline of Professor Guilliams' paper covers the following points: (1) Words do not convey ideas. (2) They are merely the symbols of ideas; the ideas must be presented objectively. (3) Sir William Hamilton's definition: "Thinking is nothing other than affirming one idea of another idea." (4) Max Müller's dictum that thought worthy of the name is impossible without language. In teaching secondary mathematics, stress should be placed in the expression in good English of every problem and demonstration. (5) Neatness and orderly arrangement of written work, with a clear statement of all the necessary data. (6) Dr. Glenn Frank's statement: "In the deepest sense, we cannot teach anybody anything; the best we can do is to help them learn for themselves." (7) The lecture method was condemned for mathematics teaching in secondary schools. Knowledge is an achievement by the learner for the learner.

He blamed the colleges for the poor work done in mathematics in the average high school of the state, stating that in some colleges of the state the only requirement for entrance was the statement that the applicant had a year's credit in algebra and a year's credit in plane geometry at Possum Trot High School, that no mathematics is required for the degree of A.B., but that when he graduates, the student can go out and teach mathematics in the high schools of the state. He urged that every mathematics teacher in the state high schools should have studied and have a good knowledge of algebra, plane and solid geometry, trigonometry, analytic geometry, and calculus, and should have had as well a careful course in the teaching of high school mathematics.

A. R. FEHN, *Secretary*

THE THIRD ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION OF THE ASSOCIATION

The third annual meeting of the Louisiana-Mississippi Section of the Mathematical Association was held at New Orleans by invitation of Newcomb College and Tulane University Friday and Saturday, March 12–13, 1926. The meeting formally opened with a dinner at six o'clock on Friday in the Newcomb Tea Room. The dinner was a splendid success and was well attended by the members and the friends of the section. President Dinwiddie of Tulane University extended a cordial welcome to all present.

Following the dinner, Professor Florian Cajori, formerly of Tulane University and now of the University of California, gave a most profitable and interesting lecture on "The history of the principles of the conservation of energy." This lecture was given before a joint session of the Section and the New Orleans Academy of Science. Besides giving this lecture and paper no. 6 which is listed below, Professor Cajori addressed the faculty and students of Louisiana State University on March 11 upon "The growth of algebraic notations," and the Faculty and students of Tulane University and Newcomb College on March 12 upon "Women students prominent in mathematical fields."

The regular program was held on Saturday morning, the chairman, Professor H. E. Buchanan, being in charge. There was a large attendance at the sessions, including the following members of the Association:

H. E. Buchanan, Julia Dall, Winifred D. Daly, W. L. Duren, Jr., Dora M. Forno, Bertha Frankenbush, J. A. Hardin, Anna M. Howe, C. G. Killen, Odessa R. Lastrapes, C. G. Latimer, A. C. Maddox, J. R. Maguire, J. E. McShane, B. E. Mitchell, I. C. Nichols, G. B. Price, S. T. Sanders, C. D. Smith, Mary C. Spencer, Norma E. Touchstone, B. A. Tucker, W. P. Webber. Resolutions were adopted expressing the sincere thanks of the members to Professor Cajori, the New Orleans Academy of Science, Louisiana State University, Tulane University, Newcomb College, and to other organizations, as well as to those in charge of the dinner. Officers elected for the year were S. T. SANDERS, Louisiana State University, Chairman; ROBERT TORREY, University of Mississippi, Vice-Chairman; B. E. MITCHELL, Millsaps College, Secretary-Treasurer. Shreveport was selected as the place for the next annual meeting.

The following papers were presented:

1. "Some series resembling the theta functions of Jacobi" by H. E. BUCHANAN, Tulane University.

2. "A fifth degree curve with certain of its related curves" by J. A. HARDIN, Centenary College.

3. "Some curves associated with the linear transformation" by B. E. MITCHELL, Millsaps College.

4. "An application of Fourier series to a problem in electricity" by E. J. McShane, Tulane University.

5. "Ceva's theorem and its application in solving certain examples given in Wentworth-Smith plane geometry" by I. C. Nichols, Louisiana State University.

6. "Origins of fourth dimension concepts" by Florian Cajori, University of California.

7. "Certain extensions of a theorem by Markoff" by C. D. Smith, Louisiana College.

8. "On certain indefinite quarternary forms representing all integers" by C. G. Latimer, Tulane University.

Abstracts of these papers are given below, the numbers corresponding to those in the list of titles.

1. In this paper Professor Buchanan discussed the functions defined by

$$Q_0(x,k)=\sum_{n=-\infty}^{\infty}(-1)^{n-1}e^{-k\rho},\quad \text{where}\quad \rho=\left(\frac{2n-1}{2}-\frac{ix}{k}\right)^4.$$

Their periodicity is established and an infinitude of zeros found.

2. Professor Hardin reported on an investigation of a one parameter curve of the fifth degree, whose equation is

$$x^5-2x^4+5ax^3y-5axy^3+2y^4=0.$$

Certain of the related curves were investigated, including the Hessian and polar curves. Particular attention was given to the relation between the curve and its Hessian, singular points of each curve, and the relation of each curve to the line at infinity.

3. The orthogonal quadrilateral arising from the dissection of the linear transformation is used as a nexus between the transformation and lines, circles conics, and higher plane curves. Since this transformation is frequently referred to as "circle transformation" Mr. Mitchell considered in particular those lines (constituting a pencil) which preserve their linear characteristic under the transformation. Again, all those circles (constituting a pencil) which lose their circle identity under the transformation. All these configurations and the others considered bear necessarily an intimate relation to the orthogonal quadrilateral which is the link between them and the linear transformation.

4. Mr. McShane discussed a certain problem in electricity wherein the apparent disagreement of theory and practice was explained.

5. The discussions by Professor Nichols were the outgrowth of his teaching of a course in Modern Geometry based upon the text by Altshiller-Court. His aim was to enlist further interest in this important study. Ceva's Theorem was stated and proved, and then its converse stated and used in proving that

(1) the medians of a triangle are concurrent, (2) the internal bisectors of a triangle are concurrent, and (3) the altitudes of a triangle are concurrent.

6. Professor Cajori began with Aristotle and traced very carefully the concept of the 4th dimension through the successive periods of scientific thought and investigation to our present time, the cravings for generalization, the efforts at occultation, the influence of the ether theory, the time concept in theoretical mechanics and in electromagnetic phenomena, and the space-time concept.

7. In treating this subject, Mr. Smith stated as a theorem of A. A. Markoff that $P \leqq 1/t^2$ where P is the probability that $U > M_1 t^2$, M_1 being the mean value of U and t an arbitrary number. By a method similar to Markoff's proof the theorem was put in the form, $P \leqq M_1/\lambda$, where P is the probability that $U > \lambda$. A geometrical interpretation of the meaning of the theorem was given and the theorem was extended to various forms.

8. In this paper it was shown that if α is a positive integer $\leqq 163$ in the form $4n+3$, or the double of such an integer, the form $x^2+y^2-\alpha z^2-\alpha w^2$ represents all integers.

I. C. NICHOLS, *Secretary*

EXTENSIONS OF WARING'S THEOREM ON NINE CUBES

By L. E. DICKSON, University of Chicago

1. Introduction. In 1770, Waring conjectured[1] that every positive integer p is the sum of the cubes of nine integers $\geqq 0$. It will be shown in §2 that three of these cubes may be taken to be equal when $p < 40{,}000$. Again, six of the cubes may be taken equal in pairs if $p < 1400$ (end of §5). These and all possible similar theorems are corollaries to

THEOREM I. *Every positive integer p can be represented by $x^3+y^3+2z^3+2u^3+3v^3$ with $x \geqq 0, \cdots, v \geqq 0$.*

This is verified in §5 for $p \leqq 1200$. It was verified by direct trial for $1200 < p \leqq 1400$.

2. THEOREM II. *Every positive integer $p < 40{,}000$ can be expressed in the form $S+3v^3$, where S is a sum of six cubes $\geqq 0$, and $v \geqq 0$.*

At the request of Jacobi,[2] the famous computer Dahse constructed a table showing the least number of positive cubes whose sum is any $p < 12000$. There are exactly 138 values of p:

(T) 7, 14, 15, 21, 22, 23, 42, $\cdots$, 5818, 8042

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[1] The first proof was given by Wieferich in 1909. For a report on the literature, see the writer's *History of the Theory of Numbers*, vol. 2, pp. 717–725.

[2] Journal für Mathematik, vol. 42 (1851), p. 41; Werke, vol. 6, p. 323.

which are not sums of six cubes $\geqq 0$. If p is a number in this table T such that $p-3$ is not in T, then $p-3$ is a sum S of six cubes, and $p=S+3$, as desired. The p's for which $p-3$ is in T are

$$50, 106, 114, 178, 295, 303, 340, 393, 1239.$$

We subtract $24=3\cdot 2^3$ from each. No difference is in T. This completes the proof[1] that every $p\leqq 12000$ can be expressed in the form $S+3v^3$ (in fact, with $v=0$, 1, or 2). By von Sterneck's[2] table showing the number of cubes needed for the representation of all numbers $p\leqq 40{,}000$, six cubes suffice when $p>8042$.

3. All forms of weight 9. The cubic form

$$(1) \qquad (a_1, \cdots, a_n) = a_1x_1^3 + \cdots + a_nx_n^3 \qquad (a_i>0,\ x_i\geqq 0)$$

shall be said to be of *order* n and *weight* $a_1+a_2+\cdots+a_n$. Since ax^3 is a sum of a cubes, a form of weight w is a sum of w cubes. Since 23 is not a sum of eight cubes, 9 is the minimum weight of a form (1) which represents all positive integers. Only when a form f is of weight 9 will expressibility of p by f imply expressibility of p as a sum of nine cubes. Hence if we seek theorems which imply Waring's theorem as a corollary, we must restrict attention to forms (1) of weight 9.

Without loss of generality, we may take $a_1\leqq a_2\leqq a_3\cdots$ in (1). Let the resulting form f represent all positive integers. Then $f=1$ implies $a_1=1$; $f=2$ implies $a_2=1$ or 2. If $a_2=1$, $f=3$ implies $a_3<4$. If $a_2=2$, $f=4$ implies $a_3<5$. Let f be of weight 9. Then $n>3$. If $n=4$, all forms are excluded:

$(1116)\neq 4$, $(1125)\neq 12$, $(1134)\neq 10$, $(1224)\neq 11$, $(1233)\neq 12$.

If $n=5$, (11223) is the only form not excluded:

$(11115)\neq 12$, $(11124)\neq 46$, $(11133)\neq 18$, $(12222)\neq 11$.

If $n=6$, the only forms are the two in Theorem III and

$$(111114)\neq 46, \quad 46 = 5\cdot 8 + 6 = 27 + 2\cdot 8 + 3.$$

THEOREM III. *If a form of weight* 9 *represents all positive integers, it is one of the following seven*:

(11223), (111222), (111123), (1111122), (1111113), (11111112), (111111111).

4. Partition process. If p is represented by (1) and if $a_1=r+s$, the first term may be replaced by $rx_1^3+sx_1^3$ to show that p is represented by $(r,s,a_2,\cdots,a_n)$. The converse usually fails since the cubes having the coefficients r and s may not be equal. Thus a proof of Theorem I would imply that every form in Theorem III represents all integers. Expressibility by any of the forms involving 3 implies expressibility by all subsequent forms. Expressibility by one not involving 3 implies expressibility by all subsequent ones not involving 3. By Theorem II, (11111112) represents all integers $<40{,}000$.

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[1] It is much shortened by assuming Theorem I, which states that every $p\leqq 1400$ is expressible as $S+3v^3$.

[2] Akademie der Wissenschaften, Wien, *Sitzungsberichte*, vol. 112, IIa (1903) pp. 1627–1666.

5. Numbers represented by $q=(1122)$ and $q+kv^3$. We employ a list[1] of the 59 numbers <1200 which are sums of two cubes $\geqq 0$. The doubles of the first forty of them are <1200, and were added to each number of the first list when the sum is <1200. After deleting the sums from $1, \cdots, 1200$, we obtain the following list[2] L of all positive integers $\leqq 1200$ which are *not* represented by q:

7, 14, 15, 21–23, 36, 38, 42, 47, 49, 50, 52, 61, 73, 75, 77, 85, 87, 92, 94, 99–103, 106, 111–5, 122, 140, 148, 150, 159, 164, 166–7, 169, 174–8, 185–6, 188, 197, 201, 204, 211–5, 223, 225, 227, 229, 230–1, 237–9, 244, 262, 276, 281, 288–90, 292, 295, 300, 302–3, 309, 311, 318, 323, 326–7, 329, 335, 337, 340, 342, 349, 356, 363–6, 390, 393, 396, 401, 403, 412, 415–7, 419–20, 422, 427–8, 431, 438, 444, 446–7, 453–4, 458, 465, 489, 491–2, 505, 507, 510, 518–9, 534, 542, 548–9, 553, 556, 570, 579, 585, 604–6, 616, 618, 627, 631, 634–5, 644, 654, 660–2, 666, 668, 670–1, 676, 678–81, 692, 699, 700–1, 705, 708, 725–7, 735, 738, 757, 759, 771, 787, 790, 796, 801, 803, 806, 820, 833, 848, 850, 852–3, 861–2, 869, 876, 882–3, 885, 888, 894–7, 913–5, 920, 922, 927, 933–5, 943, 948, 950–1, 953, 958–9, 965, 969, 970, 972, 974, 976, 988, 1009, 1011, 1014, 1021, 1044, 1047, 1050, 1063, 1065, 1077, 1092, 1100–3, 1122, 1124, 1133, 1139–40, 1148, 1167, 1174–5, 1189, 1191, 1193, 1196.

We may now easily verify that $f=q+3v^3$ represents all positive integers $p\leqq 1200$ (Theorem I). If p is not in the preceding list L, we may take $v=0$. Evidently f with $v=1$ represents p if q represents $p-3$. Hence it remains to consider the 54 numbers p of L for which $p-3$ is in L. We list them in two sets:

p = 50, 52, 102, 103, 106, 114–5, 167, 169, 177–8, 204, 214–5, 230, 292, 295, 303, 329, 340, 393, 396, 415, 419, 422, 431, 447, 492, 510, 556, 634, 671, 679, 681, 708, 738, 790, 806, 853, 888, 897, 953, 1014, 1047, 1050, 1103, 1196,

for which $p-3\cdot 2^3$ is not in L and hence is represented by q, so that p is represented by f;

$$p=188,\ 326,\ 366,\ 420,\ 885,\ 951,\ 972,$$

for which $p-24$ is in L, but $p-3\cdot 3^3$ is not in L, whence p is represented by f. This proves Theorem I for $p\leqq 1200$ (and in fact with $v\leqq 3$).

THEOREM IV. (11224) *represents all integers* $\leqq 1200$.

The thirty numbers p in list L for which $p-4$ is in L are $p=42,\ 77,\ 103,\ 106,\ 115,\ 178,\ 215,\ 227,\ 231,\ 292,\ 416,\ 419,\ 420,\ 431,\ 458,\ 553,\ 631,\ 635,\ 670,\ 680,\ 705,\ 969,\ 974,\ 976,\ 1193$, for which $p-4\cdot 2^3$ is not in L, together with $p=201,\ 229,\ 327,\ 666,\ 852$, for which $p-4\cdot 3^3$ is not in L. Hence every integer $\leqq 1200$ is represented by $q+4v^3$ with $v\leqq 3$.

But for $k\neq 3$, $k\neq 4$, $f_k=q+kv^3$ fails to represent all small integers. If $k>7$, $f_k\neq 7$. By §§6, 7, $f_2\neq 23, f_1\neq 15$. Next, $f_5\neq 47, f_6\neq 21, f_7\neq 14$.

[1] Jacobi, Journal für Mathematik, vol. 42 (1851), p. 69; Werke, vol. 6, p. 354.

[2] Rechecked with complete agreement.

By §4, Theorem I implies that (111222) represents all positive integers $\leqq 1400$. This was verified up to 712 also by adding the numbers represented by (111) to those represented by 2(111).

6. Numbers represented by $g=(11122)$ and $g+kw^3$. We first find the numbers $\leqq 1200$ represented by $g=q+v^3$. Of the 73 numbers p in list L for which $p-1$ is in L, the following have also $p-2^3$ in L:

(ϵ) 22, 50, 100, 102, 114, 186, 212, 238, 364, 420, 428, 454;

(ω) 15, 23, 167, 175, 177, 231, 239, 289, 303;

$p=635, 679, 700, 935, 951, 959, 1175$ with $p-3^3$ not in L; and $p=662$ and 896 with $p-4^3$ not in L. For every p in (ϵ) and (ω), $d=p-v^3$ is in L for all v's for which d is positive. This proves[1]

THEOREM V. *$g=(11122)$ represents all positive integers $\leqq 1200$ except those in sets (ϵ) and (ω).*

By means of this result, we shall prove

THEOREM VI. *If $k=2$–6, 10–15, (11122k) represents all positive integers $\leqq 1200$.*

First let k be odd and $\leqq 15$ (since the form evidently fails to represent 15 if $k>15$). For $w=1$, our form is $g+k$ and represents a chosen odd integer ω if the even integer $\omega-k$ is represented by g, and hence if $\omega-k$ is not one of the numbers ϵ. We find that when $\omega-\epsilon$ is positive, it exceeds 15 and hence is not equal to k except in the following two cases:

$$\omega = 23, \quad \epsilon = 22; \qquad \omega = 239, \quad \epsilon = 238 \ (\omega - \epsilon = 1).$$

The value 1 of k is excluded since no positive integer w gives $g+w^3=23$.

Similarly, $g+k$ represents a chosen even integer ϵ if the odd integer $\epsilon-k$ is represented by g, and hence if $\epsilon-k$ is not one of the numbers ω. But when $\epsilon-\omega$ is positive, it exceeds 15 except for

$$\epsilon = 22, \ \omega = 15; \quad \epsilon = 186, \ \omega = 175; \quad \epsilon = 186, \ \omega = 177; \quad \epsilon = 238, \ \omega = 231.$$

In the first and last cases, $\epsilon-\omega=7$ and $g+7w^3\neq 22$. In the second case, $g+11w^3=186$ for $w=2$. In the third case, $g+9w^3\neq 186$. The values 7 and 9 of k are therefore excluded.

Second, let k be even and $\leqq 14$. Then $g+k$ represents ϵ if the even integer $\epsilon-k$ is represented by g and hence is not an ϵ', whence $\epsilon-\epsilon'\neq k$. But the only differences $\leqq 14$ of two numbers (ϵ) are

$$102 - 100 = 2, \quad 114 - 102 = 12, \quad 114 - 100 = 14, \quad 428 - 420 = 8.$$

Similarly, $g+k$ represents ω if the odd integer $\omega-k$ is not an ω'. But the only differences $\leqq 14$ of two numbers (ω) are

$$23 - 15 = 8, \quad 175 - 167 = 8, \quad 177 - 167 = 10, \quad 177 - 175 = 2,$$
$$239 - 231 = 8, \quad 303 - 289 = 14.$$

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[1] Verified up to 800 also by addition of the numbers represented by (111) and (22). Note that the numbers stop with 454.

The value $k=8$ is excluded since $g+8w^3 \neq 23$. We see that none of

$$102 - 2\cdot 2^3 = 86,\ 114 - 12\cdot 2^3 = 18,\ 114 - 14\cdot 2^3 = 2,$$
$$177 - 10\cdot 2^3 = 97,\ 177 - 2\cdot 2^3 = 161,\ 303 - 14\cdot 2^3 = 191$$

are in (ϵ) or (ω), whence 102 is represented by $g+2w^3$, etc.

7. Numbers represented by $h=(11222)$ and $h+kw^3$. Consider $h=q+2v^3$. Of the numbers[1] p in the list L of §5 for which $p-2$ is in L, the following have also $p-2\cdot 2^3$ in L:

(ϵ) 38, 52, 166, 292, 342, 670;

(ω) 23, 77, 101, 103, 115, 213, 229, 231, 239, 365, 417;

$p=227, 311, 678, 803, 885, 1191$ with $p-2\cdot 3^3$ not in L;

$p=419, 507, 950, 974$ with $p-2\cdot 4^3$ not in L.

For every p in (ϵ) and (ω), $d=p-2v^3$ is in L for all v's for which d is positive. This proves[2]

THEOREM VII. *$h=(11222)$ represents all positive integers $\leqq 1200$ except those in sets (ϵ) and (ω).*

This is used to prove

THEOREM VIII. *If $k=1-13,\ 17-23$, $(11222k)$ represents all positive integers $\leqq 1200$.*

The proof is like that of Theorem VI. We now have the following cases: $\omega = 365$, $\epsilon = 342$, $k = 23$, $\omega - 2^3k = 181$; $\epsilon = 38, \omega = 23, g + 15w^3 \neq 38$; $\epsilon = 52, \epsilon' = 38, \epsilon - \epsilon' = 14, g + 14w^3 \neq 52$.

The only positive differences $\leqq 23$ of two ω's are

$$103 - 101 = 2,\ 115 - 101 = 14,\ 115 - 103 = 12,\ 229 - 213 = 16,$$
$$231 - 229 = 2,\ 231 - 213 = 18,\ 239 - 231 = 8,\ 239 - 229 = 10.$$

But $g+16w^3 \neq 229$. However, g represents

$$103 - 2\cdot 2^3 = 87,\ 231 - 2\cdot 2^3 = 215,\ 239 - 8\cdot 2^3 = 175,$$
$$239 - 10\cdot 2^3 = 159,\ 115 - 12\cdot 2^3 = 19,\ 231 - 18\cdot 2^3 = 87,$$

since 87, etc., are not in either (ϵ) or (ω).

8. Forms of order five. The forms $(1122k)$ were studied in §5. For $r=1$ or 3, the numbers represented by $(124r)$ were obtained from those represented by (124).

The form $f=(1234)$ represents all numbers $\leqq 800$ except 18, 22, 39, 60, 63, 74, 76, 77, 100, 103, 106–7, 117, 126, 178, 180, 201, 215, 228, 230, 245, 271, 289, 291, 295, 315, 341, 356–7, 393, 413, 419, 420, 480–1, 523, 559, 606, 611–2, 616, 671, 673, 705.

For $k=2, 3, 4, 11, 13, 14, 16, 17$, and $k>18$, the least integer not represented by $f+kv^5$ is 76, 63, 22, 74, 76, 74, 76, 39, 18, respectively. For the remaining k's, it suffices to use $v\leqq 2$ in view of the table for f. This proves

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[1] Fifty have $p-16$ not in L.

[2] Verified up to 512 also by adding the numbers represented by (11) and (222).

THEOREM IX. *(1234k) represents all positive integers* $\leqq 800$ *if and only if* $k = 1, 5$–$10, 12, 15$ *or* 18.

The form $l = (1124)$ represents all numbers $\leqq 600$ except 19, 23, 26, 38, 45, 46, 52, 53, 73, 77, 79, 90, 100–1, 103, 105, 115, 147, 166, 185, 194, 198, 202–3, 207–8, 213, 229, 231, 239, 242, 246, 268–9, 279, 292, 294, 305–6, 316, 331, 333, 335, 339, 342, 365, 368, 394, 396, 414, 416–7, 420–1, 426, 431, 446, 450, 457–8, 477, 481, 485, 487, 495, 521, 550, 583, 585–6.

For $k = 1, 4, 7, 8, 12, 13, 14, 15, 16, 17, 19$, and $k > 19$, the least integer not represented by $f + kv^3$ is 46, 23, 26, 46, 38, 90, 306, 38, 229, 90, 38, 19, respectively. For $k = 3$, it suffices to use $v \leqq 4$. For the remaining k's, it suffices to use $v \leqq 2$. This proves

THEOREM X. *(1124k) represents all positive integers* $\leqq 600$ *if and only if* $k = 2, 3, 5, 6, 9, 10, 11,$ *or* 18.

Neither $(1133k)$ nor $(1244k)$ represents all integers $\leqq 57$ for any k. If $k \geqq 8$, $r = (1248k)$ and $s = (1249k)$ fail to represent some integer $\leqq 77$ with the following exceptions: $r \neq 194$ if $k = 10$, $s \neq 315$ if $k = 12$, $s \neq 206$ if $k = 13$, $s \neq 350$ if $k = 18$.

We readily verify that, whatever be its weight, *no form of order four represents all integers* $\leqq 19$.

9. All forms of weight ten. The following are excluded:

$(111115) \neq 2 \cdot 8 + 4$, $(11116) \neq 5$, $(11125) \neq 2 \cdot 8 + 4$, $(11134) \neq 2 \cdot 8 + 2$, $(11233) \neq 39$, $(12223) \neq 39, 39 = 4 \cdot 8 + 7 = 27 + 8 + 4$.

By Theorem IV, $f = (11224)$ represents all integers $\leqq 1200$. Hence the same is true of the ten forms derived from it by partitioning one or more of 2, 2, 4 as in §4. The only further form of weight 10 and order > 4 is $g = (111133)$. By means of Jacobi's table, we obtain a list of numbers which are sums of four cubes $\geqq 0$. To each such number was added the triples of sums of two cubes. In this maner it was verified that g represents all integers $\leqq 600$.

The form $h = (1111114)$ *represents all integers* $\leqq 40{,}000$. For, if p is in table T of §2, $p - 4$ is in T only when $p = 178$, and then $p - 4 \cdot 2^3 = 146$ is not in T. The same result holds for the forms derived from h by partitioning 4.

Of the above ten forms derived from f, let l be any one except h, (111124), and (112222). If we equate to zero one of the cubes having the coefficient unity in l, we obtain a form given in Theorem III. If the latter form represents all integers, the same is evidently true of l.

10. All forms of weight eleven. The following are excluded:

$(11117) \neq 5$, $(11126) \neq 8 + 5$, $(11135) \neq 2 \cdot 8 + 7$, $(11144) \neq 8 + 7$, $(11225) \neq 47 = 5 \cdot 8 + 7 = 27 + 2 \cdot 8 + 4$, $(11333) \neq 13$, $(12224) \neq 8 + 5$, $(12233) \neq 47$, $(111116) \neq 8 + 5$, $(111125) \neq 47$, $(122222) \neq 8 + 5$, $(1111115) \neq 47$.

There remain only $F=(11234)$ of Theorem IX and fifteen forms derived from F by partitioning one or more of 2, 3, 4. Let t be any one of the fifteen except (111134), (111233) and $G=(112223)$. If we equate to zero one of the cubes in t having the coefficient unity, we obtain g or f or one of the ten forms derived from f in §9. If we omit a 2 in G, we get the form in Theorem I.

11. Conclusion. We have found numerous cubic forms which represent all positive integers less than 40,000, 1400, 1200, 800, or 600. All of minimum weight 9 are derived from (11223) by partitioning 2, 3, 3. All of weight 9 or 10 are derived similarly from (11224), (11234), or (111133). If it could be proved that any one of these four actually represents all positive integers, the same would follow for all the forms derived from it. A proof would evidently be more difficult than the proofs of Waring's theorem on nine cubes, six of which are taken to be the cubes of $a\pm u$, $a\pm v$, $a\pm w$. In any case, the theorem would require verification for numbers below some limit of considerable size. Hence the facts established in this paper have an importance beyond making our empirical theorems seem probable.

Corresponding results for fourth powers will appear in the Bulletin of the American Mathematical Society. Also a proof that all sufficiently large integers are represented by the cubic form (1, 1, 1, 1, 1, 1, 1, 2) will appear there.

The Carnegie Institution has financed the employment of assistants to aid in the elaborate investigation of fifth and higher powers.

ELEMENTARY DERIVATION OF THE FUNDAMENTAL CONSTANTS IN THE POISSON AND LEXIS FREQUENCY DISTRIBUTIONS[1]

By HAROLD T. DAVIS, Indiana University

1. Introduction. Derivations of the arithmetic average and standard deviation for frequency distributions of Poisson and Lexis type are usually made by use of theorems on mean error. The proofs of these theorems cannot be considered as strictly elementary and the underlying significance of results obtained by their use is often masked by the apparent simplicity of the proof. This has seemed to the author to be particularly true with regard to the derivation of the fundamental constants in the frequency distributions already referred to where the proof is only a matter of two or three lines when the difficulties are thrown back upon the mean error theorems.[2]

[1] Contribution 26, Waterman Institute, Indiana University.

[2] For a comprehensive treatment of Bernoulli, Poisson and Lexis series see A. Fisher: The Mathematical Theory of Probabilities, N. Y., Macmillan (1922), pp. 117–126; consult also E. Czuber: Wahrscheinlichkeitsrechnung, Teubner, Leipzig, 4th ed. (1924), vol. 1, pp. 181–183; the Handbook of Mathe-

It is the object of the present paper to derive the fundamental constants of the Poisson and Lexis distributions from the definitions in much the same way as these constants are derived in standard texts for the Bernoulli distribution.

2. Poisson Distributions. In a Poisson distribution the probability of an event varies from trial to trial within a set of m trials, but the several probabilities for one set of m trials are identical with those of every other set.

For example, suppose that we have three urns, u_1, u_2, u_3, in which we have placed white and black balls in such ratios that the probabilities of drawing a white ball from the urn are 1/3, 1/2 and 2/3 respectively. If we draw a ball from each urn we have the following eight possibilities: *BBB; WBB, BWB, BBW; WWB, WBW, BWW; WWW.* The probabilities in these cases are respectively 1/9; 1/18, 1/9, 2/9; 1/18, 1/9, 2/9; 1/9. From these values we find that the probability that we shall draw no white ball is 1/9; one white ball, 7/18; two white balls, 7/18; and three white balls, 1/9. In 72 trials we should then expect in the ideal case the following frequencies:

No. of White Balls	0	1	2	3
Frequencies	8	28	28	8

In the example, the number of white balls drawn in 72 trials is 108 or $1\frac{1}{2}$ per trial; for the square of the standard deviation we have

$$\sigma_P^2 = \{(1\tfrac{1}{2} - 0)^2 \cdot 8 + (1\tfrac{1}{2} - 1)^2 \cdot 28 + (1\tfrac{1}{2} - 2)^2 \cdot 28 + (1\tfrac{1}{2} - 3)^2 \cdot 8\}/72 = 25/36.$$

In the general case we consider n urns, $u_1, u_2, \cdots, u_n$, which contain white and black balls in such ratios that the probabilities of drawing a white ball from the urns are $p_1, p_2, \cdots, p_n$ respectively. Now let us make N drawings of n balls each, i. e. one from each urn, and record the frequencies in the cases where we shall have no white ball, one white ball, $\cdots$, n white balls. We shall thus obtain a Poisson frequency table:

No. of White Balls	0	1	$2 \cdots n$
Frequencies	f_0	f_1	$f_2 \cdots f_n$

where $f_0+f_1+f_2+\cdots+f_n=N$.

The arithmetic average, A_P, for the series is defined as the most probable value of the arithmetic average of the number of white balls drawn per trial, so that

$$A_P = (f_0 \cdot 0 + f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + \cdots + f_n \cdot n)/N.$$

We now prove the following theorem:

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matical Statistics, Cambridge, Houghton Mifflin (1924), chap. VI by H. L. Rietz; C. V. L. Charlier, Vorlesungen über die Grundzüge der mathematischen Statistik (1920), p. 30.

THEOREM I. *The arithmetic average, A_P, of the Poisson series is equal to np, where $p=(p_1+p_2+\cdots+p_n)/n$.*

PROOF: It will be convenient to adopt the following abbreviations:

$$E_0 = 1\,;\quad E_1 = p_1 + p_2 + \cdots + p_n\,;\quad E_2 = p_1p_2 + p_1p_3 + \cdots :$$
$$E_3 = p_1p_2p_3 + p_1p_2p_4 + \cdots\,;\ \cdots\,;\ E_n = p_1p_2p_3\cdots p_n.$$

We recall from algebra the following identity:

$$(x-p_1)(x-p_2)\cdots(x-p_n) \equiv E_0x^n - E_1x^{n-1} + E_2x^{n-2} - \cdots \pm E_n.$$

Since f_0 represents the number of drawings in which no white balls are obtained, we have

$$f_0 = N[(1-p_1)(1-p_2)\cdots(1-p_n)] = N[E_0 - E_1 + E_2 - \cdots \pm E_n].$$

Similarly, we have

$$f_1 = N[(1-p_1)(1-p_2)\cdots(1-p_{n-1})p_n + (1-p_1)(1-p_2)\cdots(1-p_{n-2})(1-p_n)p_{n-1} + \cdots]$$
$$= N[E_1 - 2E_2 + 3E_3 - \cdots \pm nE_n].$$
$$f_2 = N[(1-p_1)(1-p_2)\cdots(1-p_{n-2})p_{n-1}p_n + \cdots].$$

In order to express f_2 in terms of the E's, we observe that there are as many terms similar to the first one given as there are combinations of the p's taken two at a time, or ${}_nC_2$. Also, for any value of r up to $n-2$, each product of the form $(1-p_1)\,(1-p_2)\cdots(1-p_{n-2})$ has ${}_{n-2}C_r$ terms containing r letters each. There are, for example, ${}_{n-2}C_3$ terms of the form $p_1\,p_2\,p_3$. Hence, in f_2 there are altogether ${}_nC_2\cdot{}_{n-2}C_r$ terms of $r+2$ letters each. But E_{r+2}, which is the sum of products of $r+2$ letters each, contains ${}_nC_{r+2}$ terms, so that the coefficient of E_{r+2} in f_2 is equal to ${}_nC_2\cdot{}_{n-2}C_r/{}_nC_{r+2} = (r+2)!/(2!\,r!)$.

It thus follows that we have

$$f_2 = N\left[E_2 - \frac{3!}{2!1!}E_3 + \frac{4!}{2!2!}E_4 - \frac{5!}{2!3!}E_5 + \cdots \pm \frac{n!}{2!(n-2)!}E_n\right].$$

By a similar argument the other frequencies can be expressed in terms of the E's. Thus we get

$$f_3 = N\left[E_3 - \frac{4!}{3!1!}E_4 + \frac{5!}{3!2!}E_5 - \cdots \pm \frac{n!}{3!(n-3)!}E_n\right],$$

. . . . . .

$$f_n = NE_n\,.$$

From these values the arithmetic average is easily computed. Thus we have

$$\begin{aligned} A_P &= (f_0 \cdot 0 + f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + \cdots + f_n \cdot n)/N \\ &= E_1 + 2(1-1)E_2 + 3(1-1)^2 E_3 + \cdots + n(1-1)^{n-1}E_n \\ &= p_1 + p_2 + \cdots + p_n = n(p_1 + p_2 + \cdots + p_n)/n = np. \end{aligned}$$

In a similar way we can calculate the standard deviation for the Poisson series.

THEOREM II. *The square of the standard deviation, σ_P, of a Poisson series is given by the formula*:

$$\sigma_P^2 = npq - \sum_{i=1}^{n} (p_i - p)^2, \text{ where } p = (p_1 + p_2 + \cdots + p_n)/n \text{ and } q = 1 - p.$$

PROOF: Since the arithmetic average is equal to $np = E_1$, we have from the definition of the standard deviation

$$\begin{aligned} \sigma_P^2 &= [f_0(E_1 - 0)^2 + f_1(E_1 - 1)^2 + \cdots + f_n(E_1 - n)^2]/N \\ &= [(f_0 + f_1 + \cdots + f_n)E_1^2 - 2E_1 N A_P \\ &\qquad + (f_0 \cdot 0^2 + f_1 \cdot 1^2 + \cdots + f_n \cdot n^2)]/N \\ &= -E_1^2 + (f_0 \cdot 0^2 + f_1 \cdot 1^2 + \cdots + f_n \cdot n^2)/N. \end{aligned}$$

But we have

$$\begin{aligned} &(f_0 \cdot 0^2 + f_1 \cdot 1^2 + \cdots + f_n \cdot n^2)/N \\ &= E_1 - (2 - 4)E_2 + (3 - 3 \cdot 2^2 + 3^2)E_3 - \cdots. \end{aligned}$$

It will now be seen that for $r > 2$, the coefficient of E_r is equal to

$$\begin{aligned} &r \cdot 1^2 - \frac{r(r-1)}{2!}2^2 + \frac{r(r-1)(r-2)}{3!}3^2 - \cdots \pm \frac{r!}{r!}r^2 = \\ &r\left[1 - (r-1) + \frac{(r-1)(r-2)}{2!} - \cdots \pm \frac{(r-1)!}{(r-1)!}\right] \\ &\quad - r(r-1)\left[1 - (r-2) + \frac{(r-2)(r-3)}{2!} - \cdots \mp \frac{(r-2)!}{(r-2)!}\right] \\ &= r(1-1)^{r-1} - r(r-1)(1-1)^{r-2} = 0. \end{aligned}$$

It thus appears that we have

$$\sigma_P^2 = -E_1^2 + E_1 + 2E_2 = np - \sum_{i=1}^{n} p_i^2 = npq - \sum_{i=1}^{n} (p_i - p)^2.$$

3. Lexis Distributions. A Lexis distribution is characterized by the fact that the probability attached to an event is constant from trial to trial within a set, but varies from set to set.

To illustrate this definition, suppose that we have three urns, u_1, u_2, and u_3, containing black and white balls in such ratios that the probabilities fo drawing a white ball are 1/3, 1/2 and 2/3 respectively. Suppose that twenty-four drawings of three balls each, with replacements each time, are made from each of the three urns and the number of white balls recorded. The following are the probable frequencies (to the nearest integer) for each urn since each set of 24 drawings forms a Bernoulli frequency:

No. of White Balls	0	1	2	3
u_1	7	11	5	1
u_2	3	9	9	3
u_3	1	5	11	7
Totals	11	25	25	11

The totals form a Lexis distribution. The arithmetic average, A_L, and the square of the standard deviation, σ_L^2, are 3/2 and 31/36 respectively.

In the general case we have r urns, $u_1, u_2, \cdots, u_r$ containing black and white balls in such ratios that the probabilities of drawing a white ball are $p_1, p_2, \cdots, p_r$ respectively. Then N drawings of n balls each, with replacements each time, are made from each of their urns and the number of white balls is recorded. The frequency distribution for each set of N drawings for any urn is clearly a Bernoulli distribution, but the frequency distribution obtained by adding the frequencies for all of the urns will form a Lexis distribution. We can represent this as follows:

No. of White Balls	0	1	2	...	n	A	σ^2
u_1	$f_0^{(1)}$	$f_1^{(1)}$	$f_2^{(1)}$	...	$f_n^{(1)}$	np_1	np_1q_1
u_2	$f_0^{(2)}$	$f_1^{(2)}$	$f_2^{(2)}$	...	$f_n^{(2)}$	np_2	np_2q_2
...	...	...	...	...	...	...	....
u_r	$f_0^{(r)}$	$f_1^{(r)}$	$f_2^{(r)}$	...	$f_n^{(r)}$	np_r	np_rq_r
Total (Lexis Distribution)	f_0	f_1	f_2	...	f_n		

where $f_0^{(i)} + f_1^{(i)} + f_2^{(i)} + \cdots + f_n^{(i)} = N$.

We now prove the following theorem:

THEOREM III. *The arithmetic average, A_L, of the Lexis distribution is equal to np, where $p = (p_1 + p_2 + \cdots + p_r)/r$.*

PROOF:

$$A_L = (0 \cdot f_0 + 1 \cdot f_1 + 2 \cdot f_2 + \cdots + n \cdot f_n)/rN$$
$$= (A_1 + A_2 + A_3 + \cdots + A_r)/r$$
$$= n(p_1 + p_2 + p_3 + \cdots + p_r)/r = np.$$

We now calculate the standard deviation for the Lexis series.

THEOREM IV. *The square of the standard deviation, σ_L, of a Lexis series, is given by the formula*:

$$\sigma_L^2 = npq + \frac{(n^2 - n)}{r}\sum_{i=1}^{r}(p_i - p)^2, \text{ where } p = (p + p_2 + \cdots + p_r)/r \text{ and } q = 1 - p.$$

PROOF: For the calculation of the square of the standard deviation, we consider the expression:

$$\begin{aligned}\sigma_L^2 &= [f_0(np - 0)^2 + f_1(np - 1)^2 + \cdots + f_n(np - n)^2]/rN \\ &= [n^2p^2(f_0 + f_1 + \cdots + f_n)/rN] - [2np(0 \cdot f_0 + 1 \cdot f_1 + \cdots + n \cdot f_n)/rN] \\ &\quad + [(0^2f_0 + 1^2f_1 + 2^2f_2 + \cdots + n^2f_n)/rN] \\ &= -n^2p^2 + \sum_{i=1}^{r}(0^2f_0^{(i)} + 1^2f_1^{(i)} + 2^2f_2^{(i)} + \cdots + n^2f_n^{(i)})/rN.\end{aligned}$$

From the fact that

$$\sum_{j=0}^{n} f_j^{(i)}(np_i - j)^2 = Nnp_iq_i,$$

we have, after squaring and collecting terms,

$$0^2f_0^{(i)} + 1^2f_1^{(i)} + 2^2f_2^{(i)} + \cdots + n^2f_n^{(i)} = Nnp_iq_i + Nn^2p_i^2.$$

Hence we get

$$\begin{aligned}\sum_{i=1}^{r}[0^2f_0^{(i)} + 1^2f_1^{(i)} + \cdots + n^2f_n^{(i)}]/rN &= \left(\sum_{i=1}^{r} np_iq_i + n^2\sum_{i=1}^{r} p_i^2\right)/r \\ &= \left[n\sum_{i=1}^{r} p_i(1 - p_i) + n^2\sum_{i=1}^{r} p_i^2\right]/r = np + \frac{n^2 - n}{r}\sum_{i=1}^{r} p_i^2.\end{aligned}$$

Substituting this value above, we have the formula

$$\sigma_L^2 = -n^2p + np + \frac{n^2 - n}{r}\sum_{i=1}^{r} p_i^2,$$

which is easily seen to be equivalent to the expression given in the theorem.

A PERIODIC SOLUTION FOR A CERTAIN PROBLEM IN MECHANICS[1]

By J. W. CAMPBELL, University of Alberta

Suppose a mass m resting on a smooth horizontal table, is connected by a light inextensible string, which passes through a small hole in the centre of the table, to a mass M hanging freely and suppose that m is given a blow in a di-

[1] Presented to the American Mathematical Society, Sept. 9, 1926.

rection at right angles to the string on the table. Then if the centrifugal force exerted by m is equal to the weight of M, m will revolve about the hole in a circle and M will remain in stationary equilibrium. If the blow imparted to m is less than this critical value then the centrifugal force exerted by m will be less than the weight of M and M will fall. But as M falls the distance from m to the hole will decrease, the centrifugal force exerted will increase, becoming equal to and then exceeding the weight of M. The downward motion of M will thus attain a maximum value and then decrease to zero, after which M will rise. The opposite effect would be obtained if the blow imparted to m were greater than the critical value. Therefore in general M will oscillate in a vertical line.[1]

The above physical phenomonon forms the basis of problems given in Lamb's *Dynamics*, page 148, and in Byerly's *Generalised Coordinates*, page 22. The problems as there stated are only to find the total range of the motion of M, and not to find a general solution of the motion, but a general periodic solution is here obtained when the total range of motion is not too great. The method of obtaining the solution is analogous to the methods used in the finding of periodic orbits[2] in mathematical astronomy, and the main purpose of the article is to illustrate the applicability of those methods to problems of physics and mechanics where periodic variation is involved. The solution also shows some peculiarities not common to astronomical problems.

Let us denote by r the distance of m from the hole, by x the distance of M below the position from which it starts, by θ the angular position of the string on the table, and by T the tension of the string at time t.

Then by the laws of motion we obtain the equations

$$m(\ddot{r} - r\dot{\theta}^2) + T = 0$$
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$$
$$M\ddot{x} - Mg + T = 0.$$

Integrating the second of these equations and eliminating T between the first and third we obtain

$$r^2\dot{\theta} = c_1\,; \quad m(\ddot{r} - r\dot{\theta}^2) - M\ddot{x} + Mg = 0\,.$$

We now put $r = a - x$, and then, eliminating θ between these equations, we obtain

$$\ddot{x}(m + M) + mc_1^2/(a - x)^3 - Mg = 0\,.$$

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[1] It is assumed that the length of the string and the initial conditions are such that neither m nor M reach the hole before the extreme positions of M are reached.

[2] F. R. Moulton and collaborators, *Periodic Orbits*, Carnegie Publication 161.

This equation may be written in the form

$$\ddot{x} + A/(a - x)^3 - g_1 = 0,$$

$$A = mc_1^2/(m + M), \quad g_1 = Mg/(m + M). \tag{1}$$

It follows from these equations that

$$\ddot{x} = 0 \text{ when } A/(a-x)^3 = g_1, \text{ that is when } x = a - (A/g_1)^{1/3}.$$

Then put

$$x = a - (A/g_1)^{1/3} + z, \tag{2}$$

and the equation of motion becomes

$$\ddot{z} + \frac{A}{\{(A/g_1)^{1/3} - z\}^3} = g_1$$

or

$$\ddot{z} + g_1(1 - kz)^{-3} = g_1, \text{ where } k^3 = g_1/A.$$

Expanding by the binomial theorem this last equation may be written in the form

$$\ddot{z} + 3g_1k\left\{z + 2kz^2 + \frac{10}{3}k^2z^3 + 5k^3z^4 - 7k^4z^5 \cdots\right\} = 0$$

provided $z < 1/k$. The k of this equation is a parameter of the problem but it is not a convenient parameter in which to express the solution without generalisation. To obtain a periodic solution we make the transformations

$$z = cy\ ;\ 3g_1k = K\ ;\ ck = \lambda\ ;\ t - t_1 = \tau\{(1 + \delta)/K\}^{1/2} \tag{3}$$

where t_1 is the time at which $x = a - 1/k$, or at which $z = 0$ and at which dx/dt is positive.

The differential equation then becomes

$$y'' + (1 + \delta)\{y + 2\lambda y^2 + 10/3\lambda^2y^3 + \cdots\} = 0 \tag{4}$$

where the primes denote differentiation with respect to τ.

The λ is now taken as the parameter of the solution. The δ is also a parameter which is to be determined so that the motion shall be periodic with period 2π in τ, the introduction of the K as above making this possible. The introduction of the c is to facilitate the discussion and the construction of the solution, as the c may be determined so that $y'(0) = 1$.

For

$$y' = z'/c = \dot{z}\frac{dt}{d\tau}/c = \dot{z}\{(1 + \delta)/K\}^{1/2}/c,$$

since $z = cy$. Also by (2) $\dot{x} = \dot{z}$, and the energy integral of (1) gives

$$\frac{\dot{x}^2}{2} + \frac{A}{2(a - x)^2}g_1x = c_2.$$

At $x=0$, $\dot{x}=0$ and therefore $c_2=A/2a^2$. Then at $x=a-1/k$ we have

$$\dot{x}^2 = A\left(\frac{1}{a^2} - k^2\right) + 2g_1\left(a - \frac{1}{k}\right).$$

Therefore if $y'(0)=1$,

$$c = \left\{A\left(\frac{1}{a^2} - k^2\right) + 2g_1\left(a - \frac{1}{k}\right)\right\}^{1/2}\{(1+\delta)/K\}^{1/2}. \tag{5}$$

It will be convenient for later use to denote the first factor of this expression by a single symbol, and accordingly we put

$$\gamma = \left\{A\left(\frac{1}{a^2} - k^2\right) + 2g_1\left(a - \frac{1}{k}\right)\right\}^{1/2}. \tag{6}$$

Now equation (4) may be regarded as a differential equation in two parameters, δ and λ, and by the general theory of analytic differential equations it may be integrated as a power series in δ and λ, converging for δ and λ sufficiently small for any preassigned interval of time. It is true that the δ appears implicitly in λ but we shall generalise it and for the purpose of construction of the solution we shall regard it as a constant where it appears implicitly.

Let the power series solution in δ and λ be represented by

$$\begin{aligned} y &= \sum_{i,j=0}^{\infty} y_{ij}\delta^i\lambda^j \\ y_{ij}(0) &= 0 \quad (i, j = 0, \cdots, \infty) \\ y_{00}'(0) &= 1 \qquad\qquad (7) \\ y_{ij}'(0) &= 0 \quad (i, j = 0, \cdots \infty\ ;\ i+j \neq 0). \end{aligned}$$

Substituting (7) in (4) and equating coefficients of $\delta^i\lambda^j (i,j=0, \cdots \infty)$ we obtain a system of equations which may be integrated sequentially.

The coefficient of $\delta^0\lambda^0$ gives $y_{00}''+y_{00}=0$; and $y_{00}=\sin\tau$ is the solution satisfying the initial conditions stated.

The coefficient of $\delta\lambda^0$ gives $y_{10}''+y_{10}=-\sin\tau$.

The solution of this equation satisfying the initial conditions is readily found to be

$$y_{10} = -\frac{1}{2}\sin\tau + \frac{\tau}{2}\cos\tau.$$

These two steps are sufficient to establish the existence of a periodic solution of (4). For the conditions of such a solution with period 2π in τ are

$$y(2\pi) = y(0) = 0\ ;\ y'(2\pi) = y'(0) = 1. \tag{8}$$

These conditions are not independent however, as the first implies the second. For since $y(0)=0$ and $y'(0)=1$, the energy integral of (4) implies that if $y(2\pi)=0$, then

$$\{y'(2\pi)\}^2 = \{y'(0)\}^2 = 1, \quad \text{and therefore} \quad y'(2\pi) = \pm 1.$$

But since by the power series solution just obtained

$$y = \sin\tau + \left(-\tfrac{1}{2}\sin\tau + \frac{\tau}{2}\cos\tau\right)\delta + \cdots,$$

it follows that

$$y(2\pi) = \pi\delta + q(\delta,\lambda), \quad q(0,0) = 0, \tag{9}$$

$$y'(2\pi) = 1 + p(\delta,\lambda), \quad p(0,0) = 0. \tag{10}$$

Equation (9) may be solved for δ as a power series in λ, vanishing with λ and converging for λ sufficiently small. Also (10) is true for λ sufficiently small. But when $\lambda=0$, by (10) $y'(2\pi)=1$. Therefore $y'(2\pi)=+1$, and not -1, for λ sufficiently small. Therefore the first of equations (8) implies the second.

Again changing the signs of y, λ and τ leaves the differential equation (4) unchanged and also the initial conditions $y(0)=0$, $y'(0)=1$ unchanged. Therefore if $y=f(\lambda, \delta; \tau)$ is a solution, $-y=f(-\lambda, \delta; -\tau)$.

And if $f(\lambda, \delta; \tau)$ is periodic with period 2π, then $f(-\lambda, \delta; -\tau)$ is also periodic with period 2π. Hence if δ is determined so that $f(\lambda, \delta; 2\pi)=0$, then

$$f(-\lambda, \delta; -2\pi) = f(-\lambda, \delta; 2\pi) = 0.$$

This means that $f(\lambda, \delta; 2\pi)$ is explicitly even in λ and therefore δ is expansible as a power series in even powers of λ.

Let such an expansion be represented by

$$\delta = \delta_2\lambda^2 + \delta_4\lambda^4 + \delta_6\lambda^6 + \cdots. \tag{11}$$

Now by (3), (5) and (6), $\lambda^2 = c^2k^2 = \gamma^2k^2(1+\delta)/K$,

and if we put $\mu = \gamma^2k^2/K$, then

$$\delta = \delta_2\mu(1+\delta) + \delta_4\mu^2(1+\delta)^2 + \delta_6\mu^3(1+\delta)^3 + \cdots. \tag{12}$$

But this is again an implicit equation in δ and μ which may be solved for δ as a power series in μ, vanishing with μ and converging for μ sufficiently small.

The solution is

$$\delta = \delta_2\mu + (\delta_2^2 + \delta_4)\mu^2 + (\delta_2^3 + 3\delta_2\delta_4 + \delta_6)\mu^3 + \cdots.$$

With this value for δ the solution (7) is periodic.

Direct Construction of the Solution

Instead of finding $f(\lambda, \delta; \tau)$ in the manner indicated and solving $f(\lambda, \delta; 2\pi)=0$ for δ to get equation (11), it is simpler to solve equation (4) directly as a power series in λ, determining the coefficients of the expansion (11) at each step so that the solution shall be periodic with period 2π in τ.

Accordingly we assume the solution

$$y = \sum_{i=0}^{\infty} y_i \lambda^i ; \qquad \delta = \sum_{i=1}^{\infty} \delta_{2i} \lambda^{2i}$$

$$y_i(0) = 0 \ (i = 0, \cdots, \infty)$$

$$y_0'(0) = 1, \quad y_i'(0) = 0 \ \ (i = 1, \cdots, \infty).$$

This solution must satisfy the equation identically in λ, and therefore after the substitution we may equate the coefficients of λ^i separately to zero. There results a system of differential equations which may be integrated sequentially, the constants of integration being determined at each step so as to satisfy the above conditions.

The coefficient of λ^0 gives $y_0'' + y_0 = 0$. The solution is $y_0 = \sin\tau$.

The coefficient of λ gives $y_1'' + y_1 = -2y_0^2 = -1 + \cos 2\tau$.

The solution of this equation is

$$y_1 = -1 + \frac{4}{3}\cos\tau - \frac{1}{3}\cos 2\tau.$$

The coefficient of λ^2 gives

$$y_2'' + y_2 = -\left(4y_0y_1 + \frac{10}{3}y_0^3 + \delta_2 y_0\right)$$

$$= \left(-\delta_2 + \frac{5}{6}\right)\sin\tau - \frac{8}{3}\sin 2\tau + \frac{3}{2}\sin 3\tau.$$

For periodicity, $\delta_2 = 5/6$, and the solution of the equation gives

$$y_2 = -\frac{175}{144}\sin\tau + \frac{8}{9}\sin 2\tau - \frac{3}{16}\sin 3\tau.$$

The coefficient of λ^3 gives

$$y_3'' + y_3 = -\left(2y_1^2 + 4y_0y_2 + 10y_0^3y_1 + 5y_0^4 + \frac{5}{6}y_1 + \frac{5}{3}y_0^2\right)$$

$$= \frac{5}{6} - \frac{44}{9}\cos 2\tau + 6\cos 3\tau - \frac{35}{18}\cos 4\tau.$$

The solution of this equation is

$$y_3 = \frac{5}{6} - \frac{199}{108}\cos\tau + \frac{44}{27}\cos 2\tau - \frac{3}{4}\cos 3\tau + \frac{7}{54}\cos 4\tau.$$

This method may be carried on indefinitely and as many of the coefficients as desired obtained. The coefficients of odd powers of λ are cosine series in multiples of τ and the coefficients of even powers are sine series. It seems rather remarkable that while the determination of a δ_{2i} is necessary in order to eliminate the term in $\sin\tau$ in obtaining the coefficients of even powers of λ, in obtaining the coefficients of odd powers the term in $\cos\tau$ drops out automatically. There seems to be no simple direct proof of this fact, but that it is true follows from the existence and the uniqueness of the periodic solution.

Also, the values of the coefficients obtained may be checked by means of the energy integral of equation (4). That integral, after simplification, is

$$y'^2 + a\ \text{constant} = -(1+\delta)\left(y^2 + \frac{4}{3}\lambda y^3 + \frac{5}{3}\lambda^2 y^4 + 2\lambda^3 y^5 + \frac{7}{3}\lambda^4 y^6 + \cdots\right).$$

This equation must be satisfied identically in λ, and therefore the substitution in it of the solution (12) requires that the following relations be satisfied.

$$y'^2_0 + a\ \text{constant} = y_0^2$$

$$2y_0' y_1' = -\left(2y_0 y_1 + \frac{4}{3} y_0^3\right)$$

$$y_1'^2 + 2y_0' y_2' = -\left(y_1^2 + 2y_0 y_2 + 4y_0^2 y_1 + \frac{5}{3} y_0^4 + \delta_2 y_0^2\right)$$

$$2y_1' y_2' + 2y_0' y_3' = -\left\{2y_1 y_2 + 2y_0 y_3 + 4y_0^2 y_2 + 4y_0 y_1^2 + \frac{20}{3} y_0^3 y_1 + 2y_0^5 + \delta_2\left(2y_0 y_1 + \frac{4}{3} y_0^3\right)\right\}$$

. . . . . .

Finally, the solution in terms of the original data is, on substitution,

$$x = a - \frac{1}{k} + c\left\{\sin\tau - \frac{1}{3}(3 - 4\cos\tau + \cos 2\tau)\lambda - \frac{1}{144}(175\sin\tau - 128\sin 2\tau + 27\sin 3\tau)\lambda^2 + \frac{1}{108}(90 - 199\cos\tau + 176\cos 2\tau - 81\cos 3\tau + 14\cos 4\tau)\lambda^3 + \cdots\right\}$$

where

$$\tau = (t - t_1)/\{(1 + \delta)/K\}^{1/2}; \quad \delta = 5/6\mu + \cdots$$

$$c^2 = \gamma^2(1 + \delta)/K; \quad \lambda = ck; \quad \mu = \gamma^2 k^2/K$$

$$\gamma^2 = A\left(\frac{1}{a^2} - k^2\right) + 2g_1\left(a - \frac{1}{k}\right); \quad A = \frac{ma^4\dot{\theta}_0^2}{m + M},$$

$$g_1 = \frac{Mg}{m + M}; \quad k = (g_1/A)^{1/3}; \quad K = 3g_1k.$$

The general solution shows that the motion is non-symmetrical about the equilibrium position.

To illustrate by a numerical example, if we take $a = 10$ ft., $m = 0.1$ lb., $M = 2.9$ lb., $a\dot{\theta}_0 = 100$ ft./sec. and $g = 32.16$ ft./sec^2., then

$$A = 33333 \text{ ft.}^4/\text{sec.}^2 \qquad \mu = 0.0005465$$

$$g_1 = 31.088 \text{ ft./sec.}^2 \qquad \delta = 0.000455$$

$$k = 0.097702 \text{ ft.}^{-1} \qquad c = 0.239 \text{ ft.}$$

$$= 9.1120\ 8 \text{ sec.}^{-2} \qquad \lambda = 0.0234$$

$$\gamma^2 = 0.5217 \text{ ft.}^2/\text{sec.}^2$$

Then the period is

$$P = \{(1 + \delta)/K\}^{1/2}2\pi = 2.08 \text{ sec.}$$

Solution by interpolation shows that the motion below the equilibrium position requires 1.02 sec. per oscillation and that above 1.06 sec. The total range of oscillation is 0.478 ft., of which 0.235 ft. is below the equilibrium position and 0.243 ft. is above.

SUGGESTED READINGS IN CONNECTION WITH "SUCCESSIVE GENERALIZATIONS IN THE THEORY OF NUMBERS"[1]

By E. T. Bell, California Institute of Technology

R. Dedekind. *Sur la théorie des nombres entiers algébriques*, Bulletin des Sciences Mathématiques (Darboux' Bulletin), (2), vol. 1, pp. 17–41, 69–92, 114–164, 207–248. Also published separately by Gauthier Villars, Paris, 1877.

——— *Vorlesungen über Zahlentheorie*, von P. G. Lejeune Dirichlet, 4th edition, 1894 (reprinted in 1912), Supplement XI.

G. Zolotareff. *Théorie des nombres entiers complexes, avec une application au calcul intégral*. St. Petersbourg, 1873–4. Chapter III for theory of ideals.

——— Journal des Mathématiques (Liouville's Journal), (3), vol. 6, (1880) pp. 51–84, 115–129, 145–167.

[1] This paper by Professor E. T. Bell was published in the February number of this Monthly, vol. 34 (1927), pp. 55–75.

E. Landau. *Einführung in die elementare und analytische Theorie der algebräischen Zahlen und der Ideale*, 1918, Teubner. Especially pp. 1–54.

E. Lasker. Mathematische Annalen, 60, 1905, pp. 20–115, 607–8.

P. Bachmann. *Zahlentheorie. V, Allgemeine Arithmetic der Zahlenkörper.* Leipzig, 1905.

K. Hensel. *Zahlentheorie* (Göshen, 1913).

L. E. Dickson. Bulletin of the American Mathematical Society, vol. 13 (1907), pp. 348–362. Annals of Mathematics, (2), vol. 18 (1917) pp. 161–187.

H. Prüfer. *Neue Begründung der algebräischen Zahlentheorie*, Mathematische Annalen, vol. 94 (1925), pp. 198–243.

A. Speiser. *Allgemeine Zahlentheorie.* Vierteljahrschrift der Naturforschenden Gesellschaft zu Zürich, vol. 71 (1926), pp. 1–48.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. THE COMPOSITION OF ANGULAR VELOCITIES.

By F. E. KESTER, University of Kansas

Perhaps the most convenient and elegant treatment of the theorem for the composition and resolution of angular velocities for class use consists in showing that such quantities are completely represented by vectors, invoking then the general principle of vector composition and resolution. There is, however, excuse for reasonable doubt, it has always seemed to me, that a body can follow strictly the dictates of two or more simultaneous angular velocities without some distortion of form—doubt, in other words, that a *rigid* body can be subjected simultaneously to more than one angular velocity.

Such doubt as this is not quieted by the usual treatment found in text books on mechanics designed for intermediate courses, say for students of a senior college. Books of this grade, when they deign to treat the theorem, usually limit the considerations to such points of the body as lie in the plane of the axes (concurrent) of two given angular velocities.[1] The question as to distortions of the body, if the above mentioned doubt exists at all, regards more seriously those points which are out at more general positions in the body.

The only complete treatments of which I know make use of the Euler transformation of coordinates (Thomson and Tait, Webster, Whittaker, and

[1] The only extension of the usual treatment that I have been able to find is in Gray & Gray, *Treatise on Dynamics* (1911), pp. 448–9.

similar treatises), and these are more formidable than one cares to weave into an undergraduate course in mechanics. Sometimes, moreover, the theorems so given are somewhat misleadingly stated in terms of the composition of very small angular displacements.

The following presentation of this theorem has been used in my courses in mechanics over a number of years. I first stated it entirely in terms of direction cosines and sines; recently however, in class, I inadvertently started with expressions for the component velocities (linear) of a general particle of the body which is subjected to the given angular velocities, in terms of the coordinates of that particle, and I was much surprised, on following this lead through, to find how much simpler the formulation became and how much more evident the conclusion. Thinking that the theorem in its present form may prove useful to others who may have experienced the same desire to give the proof in complete yet simple manner, I am publishing it here for general distribution.

Assume that the body in question is subjected simultaneously to three angular velocities, w_x, w_y, w_z, about three concurrent and mutually perpendicular axes. Choose for consideration any particle P of the body, distant r from O, the point of intersection of the axes; the coordinates of P are x, y, z, (Fig. 1).

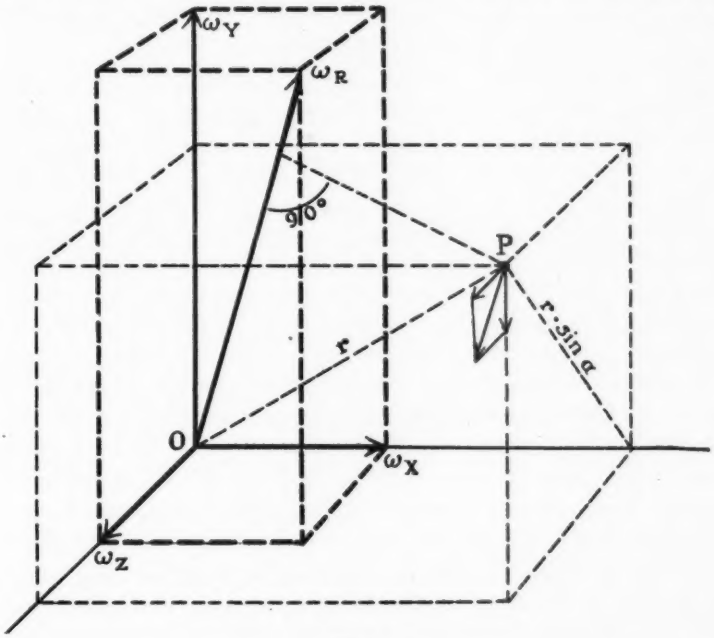

FIG. 1.

If α, β, γ, are the angles which OP makes with the three axes, the linear velocity of P due to w_x is $w_x r \cdot \sin\alpha$, in a plane perpendicular to the x-axis. The x-component of this velocity is of course zero; the y-component is $-w_x r \cdot \sin\alpha(z/r\sin\alpha)$, or $-w_x z$; the z-component is $+w_x y$. We may express the various linear velocity components in tabular form:

Velocity component along	x	y	z
due to w_x	0	$-w_x z$	$+w_x y$
due to w_y	$+w_y z$	0	$-w_y x$
due to w_z	$-w_z x$	$+w_z y$	0

Then the resultant linear velocity of P is given by

$$\begin{aligned} v^2 &= (w_y z - w_z y)^2 + (w_z x - w_x z)^2 + (w_x y - w_y x)^2 \\ &= w_x^2(y^2 + z^2) + w_y^2(z^2 + x^2) + w_z^2(x^2 + y^2) - 2w_x w_y xy - 2w_y w_z yz - 2w_z w_x zx \\ &= (w_x^2 + w_y^2 + w_z^2)r^2 - w_x^2 x^2 - w_y^2 y^2 - w_z^2 z^2 - 2w_x w_y xy - 2w_y w_z yz - 2w_z w_x zx \\ &= (w_x^2 + w_y^2 + w_z^2)r^2 - (w_x x + w_y y + w_z z)^2 \\ &\equiv (w_x^2 + w_y^2 + w_z^2)\left[r^2 - \left(x\frac{w_x}{(w_x^2 + w_y^2 + w_z^2)^{1/2}}\right.\right. \\ &\qquad \left.\left. + y\frac{w_y}{(w_x^2 + w_y^2 + w_z^2)^{1/2}} + z\frac{w_z}{(w_x^2 + w_y^2 + w_z^2)^{1/2}}\right)^2\right]. \end{aligned}$$

The quantity $(x \cdot w_x/(w_x^2+w_y^2+w_z^2)^{1/2}+\cdots)$ may be considered as the projection of r, through its components x, y, z, on to a line whose direction cosines are $w_x/(w_x^2+w_y^2+w_z^2)^{1/2}, \cdots$, so that

$v^2 = (w_x^2 + w_y^2 + w_z^2)(r^2 - r^2\cos^2\theta)$, and $v = r\sin\theta(w_x^2 + w_y^2 + w_z^2)^{1/2}$, where θ is the angle between r and the above mentioned line.

This expression for v may be interpreted as the result of a single angular velocity of magnitude $(w_x^2+w_y^2+w_z^2)^{1/2}$ about an axis the direction cosines of which are $w_x/(w_x^2+w_y^2+w_z^2)^{1/2}$; $w_y/(w_x^2+w_y^2+w_z^2)^{1/2}$; $w_z/(w_x^2+w_y^2-w_z^2)^{1/2}$; and this resultant angular velocity may be represented by the diagonal of the rectangular parallelopipedon whose adjacent edges are w_x, w_y, w_z.

Inasmuch as P is any general particle within the body, it is evident that the whole body moves as though subjected to only one angular velocity, therefore, also that a rigid body may be subjected strictly to two or more angular velocities.

By the converse of the theorem thus far proved we may conclude that any given angular velocity may be resolved into components along mutually perpendicular axes. By such resolution of, say, two given angular velocities ω_1 and ω_2 with axes concurrent and in general directions, and by subsequent composition of the components thus obtained, it becomes easy to prove geometrically that the resultant ω_R of the two angular velocities (with concurrent axes) is given by the adjacent diagonal of the parallelogram plotted with the two given vectors as sides. Finally we may proceed to the more complete theorem of the polygon of angular velocities—which brings us back inductively to the treatment, first mentioned, of vector addition (and subtraction).

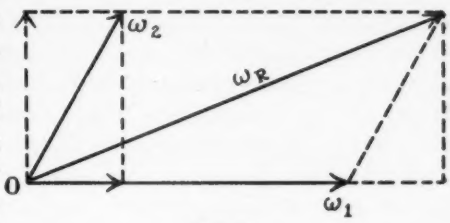

FIG. 2.

II. THE DEVIL'S CURVE AND ABELIAN INTEGRALS

By PAUL R. RIDER, Washington University

1. The devil's curve. The curve

$$y^4 - x^4 + ay^2 + bx^2 = 0 \tag{1}$$

is called by French geometers *la courbe du diable*.[1] The origin of this satanic appellation seems to be unknown, but B. H. Brown[2] suggests that "in all probability the curve was summarily christened by some exasperated youth who felt strongly and expressed himself thus forcibly on the subject." Professor Brown gives in his note a bibliography, to which the reader is referred.

The principal interest in the curve seems to be that it affords a good exercise in curve tracing—discussion of symmetry, determination of horizontal and vertical tangents, asymptotes, singular points, etc. Professor Pierpont once called the attention of the writer to the fact that it is also an excellent example to be used in presenting the theory of Riemann surfaces and Abelian integrals.

The form of the curve most frequently met is

$$y^4 - x^4 - 96ay^2 + 100a^2x^2 = 0, \tag{2}$$

a special case of (1). The special case which we wish to consider here is

$$y^4 - x^4 - y^2 + 2x^2 = 0, \tag{3}$$

[1] See Loria, *Spezielle algebraische und transzendente ebene Kurven* (1902), p. 97, footnote 2 or Zweite Auflage (1910), p. 101, footnote 2. See also Cramer, *Introduction à l'Analyse des Lignes Courbes Algébriques* (1750), p. 19ff. Loria refers to L'Intermédiaire des Mathématiciens, vol. 4 (1897); the page number, which Loria gives incorrectly as 222, should be 104. In this place Brocard lists (without definition) a large number of curves which have received special names; among them is *courbe du diable*.

[2] B. H. Brown, *La courbe du diable*, this MONTHLY, vol. 33 (1926), pp. 273–274.

results for which have been worked out by the writer on a previous occasion. The general characteristics of curve (3) are the same as those of curve (2). Curve (1) has a node or an isolated point at the origin according as $ab \lesseqgtr 0$; consequently (2) and (3) have nodes at the origin. Results for (3) are merely summarized; much of the work is too elementary to be given in detail, and for any points of theory concerning which the reader desires further information he may consult Landfriedt, *Theorie der algebraischen Funktionen und ihrer Integrale*, or Appell and Goursat, *Théorie des Fonctions Algébriques et de leurs Intégrales*. Other interesting features of the curve will doubtless be found by a more extensive investigation.

Singular point (node) at (0, 0).

Horizontal tangents at (0, 1), (0, −1).

Vertical tangents at $(\sqrt{2},0),(-\sqrt{2},0),(\frac{1}{2}\sqrt{3}+\frac{1}{2},\frac{1}{2}\sqrt{2}),(\frac{1}{2}\sqrt{3}-\frac{1}{2},\frac{1}{2}\sqrt{2})$, $(-\frac{1}{2}\sqrt{3}+\frac{1}{2},\frac{1}{2}\sqrt{2})$, $(-\frac{1}{2}\sqrt{3}-\frac{1}{2},\frac{1}{2}\sqrt{2})$, $(\frac{1}{2}\sqrt{3}+\frac{1}{2},-\frac{1}{2}\sqrt{2})$, $(\frac{1}{2}\sqrt{3}-\frac{1}{2},-\frac{1}{2}\sqrt{2})$, $(-\frac{1}{2}\sqrt{3}+\frac{1}{2},-\frac{1}{2}\sqrt{2})$, $(-\frac{1}{2}\sqrt{3}-\frac{1}{2},-\frac{1}{2}\sqrt{2})$.

Asymptotes: $y=x, y=-x$ (also imaginary asymptotes $y=ix, y=-ix$).

Plücker's formulas:

$m=$ degree of curve $=4$,

$d'=$ number of possible double points $=(m-1)(m-2)/2=3$,

$d=$ number of actual double points $=1$,

$p=d'-d=$ genus or deficiency $=2$,

$k=$ number of cusps $=0$,

$c=$ class $=m(m-1)-2d-3k=10$,

$i=$ number of inflection points $=3m(m-2)-6d-8k=18$,

$t=$ number of double tangents $=\frac{1}{2}c(c-1)-m-3i=16$.

2. Riemann surface. The values of x for which $F(x,y)\equiv y^4-x^4-y^2+2x^2=0$, has multiple roots in y are $x=0,\ \sqrt{2},\ -\sqrt{2},\ \frac{1}{2}+\frac{1}{2}\sqrt{3},\ -\frac{1}{2}+\frac{1}{2}\sqrt{3},\ \frac{1}{2}-\frac{1}{2}\sqrt{3},\ -\frac{1}{2}-\frac{1}{2}\sqrt{3},\ \infty$.

Developments about various points:

At $(0,0)$, $y=\pm 2^{1/2}x+\cdots$.

At $(\sqrt{2},0)$, $y=\pm 2^{5/4}i(x-\sqrt{2})^{1/2}+\cdots$.

Similar developments to those at $(\sqrt{2},0)$ hold at $(-\sqrt{2},0)$.

At $(\frac{1}{2}+\frac{1}{2}\sqrt{3},\frac{1}{2}\sqrt{2})$, $y=\frac{1}{2}\sqrt{2}\pm i(\frac{3}{2}+\frac{1}{2}\sqrt{3})^{1/2}(x-\frac{1}{2}-\frac{1}{2}\sqrt{3})^{1/2}+\cdots$.

Similar developments to these are found at the other multiple roots (except ∞).

At (∞,∞), $1/y=\pm 1/x+\cdots, 1/y=\pm i/x+\cdots$.

In obtaining these developments one may illustrate Puiseux's method and call attention to Newton's parallelogram amd De Gua's triangle.

The branch points are therefore $x_1=\sqrt{2}$, $x_2=-\sqrt{2}$, $x_3=\frac{1}{2}+\frac{1}{2}\sqrt{3}$, $x_4=-\frac{1}{2}+\frac{1}{2}\sqrt{3}$, $x_5=\frac{1}{2}-\frac{1}{2}\sqrt{3}$, $x_6=-\frac{1}{2}-\frac{1}{2}\sqrt{3}$.

If we solve equation (3) for y we find that

$$y_1=(\tfrac{1}{2}+r)^{1/2},\ y_2=(\tfrac{1}{2}-r)^{1/2},\ y_3=-(\tfrac{1}{2}+r)^{1/2},\ y_4=-(\tfrac{1}{2}-r)^{1/2},$$

where $r=(x^4-2x^2+\frac{1}{4})^{1/2}=[(x-x_3)(x-x_4)(x-x_5)(x-x_6)]^{1/2}$.

Let us form a Riemann surface by joining by branch cuts x_1 and x_2, x_3 and x_5, x_4 and x_6. Then the branches permute as shown in the following scheme:

Point	Connection of sheets	Order of Cycles
x_1	13 (24)	$0+0+1=1$
x_2	13 (24)	$0+0+1=1$
x_3	(12)(34)	$1+1=2$
x_4	(12)(34)	$1+1=2$
x_5	(12)(34)	$1+1=2$
x_6	(12)(34)	$1+1=2$

Thus the sum of the orders of all cycles is 10, and by means of the well-known formula $p=(\frac{1}{2})$(sum of orders of cycles)$-(m-1)$ we find that $p=5-3=2$, which checks our previous result.

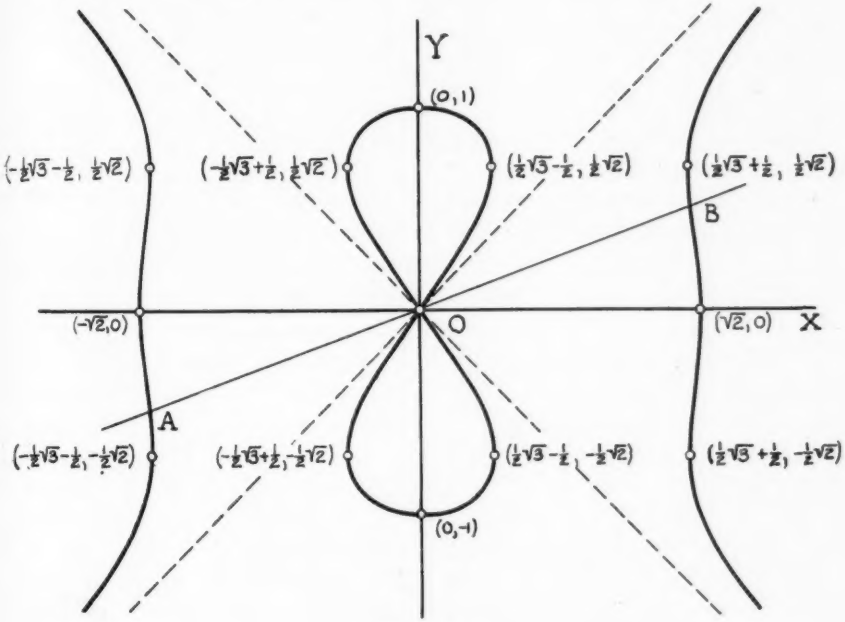

FIGURE 1

3. **Abelian integral of first species.** To form an Abelian integral of the first species belonging to (3) let us take the adjoint polynomial of degree $m-3=1$, which, since the curve represented by it must pass through the double point $(0,0)$, is of the form cx. The line $y-cx=0$ is the line AB in Figure 1 and the integral

$$I_1 = \int \frac{y - cx}{F_y} dx = \int \frac{(y - cx)dx}{2y(2y^2 - 1)}$$

is an Abelian integral of the first species, being finite at all points of the devil's curve.

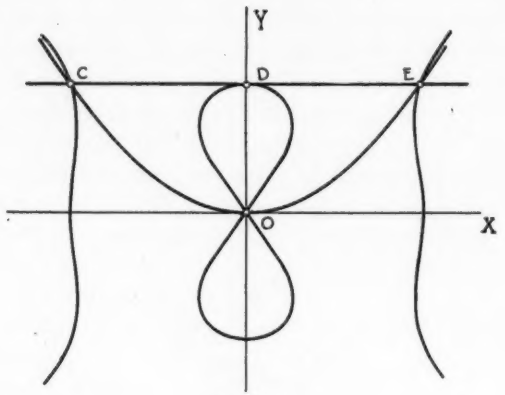

FIGURE 2.

4. Abelian integral of second species. Let us take the line $y-1=0$, which is tangent to the devil's curve at $D(0,1)$ and cuts it in the other points $C(-\sqrt{2}, 1)$, $E(\sqrt{2},1)$. (See Figure 2.) The adjoint of degree $m-2=2$ which passes through C and E is $y-(c_1x^2+c_2x)=y-\frac{1}{2}x^2=0$. Then

$$I_2 = \int \frac{y - \frac{1}{2}x^2}{(y-1)F_y} dx = \int \frac{(y - \frac{1}{2}x^2)dx}{(y-1)(4y^3 - 2y)}$$

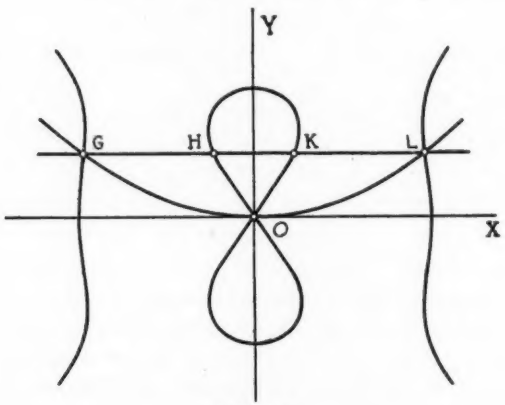

FIGURE 3.

is an Abelian integral of the second species. It has a pole of order 1 at the point D, but is finite elsewhere on the devil's curve.

5. Abelian integral of third species. Take the line $y-\frac{1}{2}=0$, which cuts the devil's curve in the points $G[-(1+\frac{1}{4}\sqrt{13})^{1/2}, \frac{1}{2}]$, $H[-(1-\frac{1}{4}\sqrt{13})^{1/2}, \frac{1}{2}]$, $K[(1-\frac{1}{4}\sqrt{13})^{1/2}, \frac{1}{2}]$, $L[(1+\frac{1}{4}\sqrt{13})^{1/2}, \frac{1}{2}]$ of Figure 3. The adjoint of degree $m-2=2$ which passes through G and L is $y-\frac{1}{3}(8-2\sqrt{13})x^2=0$. Then

$$I_3=\int \frac{y-\frac{1}{3}(8-2\sqrt{13})x^2}{(y-\frac{1}{2})F_y}dx=\int \frac{y-\frac{1}{3}(4-\sqrt{13})x^2}{(y-\frac{1}{2})(2y^3-y)}dx$$

is an Abelian integral of the third species, as it can readily be shown that its only singularities are at H and K and that these are logarithmic.

III. Note on the Function $y=a^x$, $a<0$.

By Alan D. Campbell, University of Arkansas.

No mention has been found by the author anywhere concerning the peculiarities of the function of x defined by $y=a^x$, $a<0$. If we try to plot the graph of this function we find how discontinuous it is. Thus $a^{\pm 1/2}$ is imaginary, $a^{\pm 1/3}$ is real and negative, $a^{\pm 2/3}$ is real and positive, $a^{\pm 2}$ is positive, $a^{\pm 3}$ is negative. We cannot define uniquely the value of $y=a^x$, $a<0$ for $x=\sqrt{2}$; because if we let $x\to\sqrt{2}$ thru the sequence 1.41, 1.414, $\cdots$ we get some imaginary values for a^x, but if we let $x\to\sqrt{2}$ thru the sequence 141/99, 1414/999, $\cdots$ we get real values for a^x, some positive and some negative.

Again $a^x/a^x=1$; but $\lim_{x\to 0}a^x=-1$ if $x\to 0$ thru such a sequence as $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\cdots$, whereas if $x\to 0$ thru $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\cdots$, a^x takes on imaginary values; and a^x takes on both positive and negative real values as $x\to 0$ thru the sequence $\frac{2}{3}, \frac{1}{3}, \frac{2}{9}, \frac{1}{9}, \frac{2}{27}, \frac{1}{27}, \cdots$; finally a^x takes on both real and imaginary values as $x\to 0$ thru the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots$. If we endeavor to define a derivative for $y=a^x$, $a<0$, by the same method as when $a>0$, we find that the derivative does not exist. The same is true of the integral. For a complete discussion of $y=a^x$, $a<0$, recourse should be had to complex variables. In this note we are looking at this function from the viewpoint of real variables. It seems as though $y=a^x$, $a<0$, might be used along with such functions as $y=\sin(1/x)$, $y=x\sin(1/x)$, etc., to introduce students to the singularities of real functions.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

THE DUHAMEL THEOREM

A DISCUSSION OF A REVIEW OF WOODS' "ADVANCED CALCULUS," IN THREE PARTS.

By F. S. WOODS, W. F. OSGOOD, W. B. CARVER.

I. A LETTER FROM F. S. WOODS TO THE EDITOR-IN-CHIEF.

Dear Sir:

In a review of my Advanced Calculus in the January number of the American Mathematical Monthly, Professor H. G. Ettlinger takes exception to my statement and use of Duhamel's theorem. Since the matter is one of great mathematical and pedagogical interest to all teachers of the Calculus, I hope you will give me an opportunity to reply. This is all the more desirable because Professor W. F. Osgood, the weight of whose authority is thrown against me by the reviewer, in reality supports my position.

The reviewer says, "In 1903 W. F. Osgood showed that this theorem is false," and then proceeds to name certain authors who have nevertheless used the theorem since. The implication is that these authors are ignorant of Osgood's work. But the reviewer might have added to his list the name of W. F. Osgood who uses the theorem in his text on the Calculus published in 1907 although he modified the statement later, in 1922. Hence we have the astounding implication, according to Ettlinger, that Osgood is ignorant of his own work.

As a matter of fact what Osgood did in 1903 was not to prove the theorem false but to show explicitly the conditions under which it is true. These are that $\lim \epsilon = 0$ and that $\lim \alpha_i/\beta_i = 1$ no matter how i varies with n. Both conditions are met in my book. The first one is part of the very definition of "principal part" and the only limits considered are those which are independent of the method of approach. Possibly the latter assumption might have been explicitly stated, but that is unnecessary for the students for whom the book is written and would only serve to satisfy the critic.

My choice of Duhamel's theorem in the form in which I have given it was deliberately made on pedagogical grounds. My students are physicists and engineers. They have been constantly using Duhamel's theorem without knowing it, and the authors they read in applied mathematics are doing the same thing. It is essential that the student should be taught what is involved in such work. That it may be difficult to prove in a specific case that the infinitesimals involved satisfy the conditions of the theorem is beside the point. It is important that the student should know that such a proof is necessary, and that if it is neglected, as it often is, his results are open to doubt.

The various modifications of Duhamel's theorem, so pleasing to some mathematicians, would be of no interest to these students and would have no contact with their work elsewhere. I might enlarge on this point if the matter were not so finely put in the following letter from Professor Osgood, who has given me an unsolicited permission to publish it. His postscript has reference to another criticism in the Ettlinger review the point of which I fail to grasp.

F. S. Woods, Massachusetts Institute of Technology

II. A Letter from W. F. Osgood to F. S. Woods

Dear Woods:

The trouble with Ettlinger's review is that it fails to recognize what, to the student, truth is. I have no quarrel with you for your statement of Duhamel's theorem, p. 22 (the words "and each infinitesimal approaches zero" seem superfluous, since an infinitesimal has been defined as a variable which approaches zero, but that is not what is worrying Ettlinger). In fact, it is precisely the form I had used in the first edition of the Calculus, and I modified it in the later edition, not because I believed the revised form would be more useful to the student, but to forestall carping criticism.

I recall vividly my own experience as a student with Duhamel's theorem, in the simplified form above referred to. It meant a great deal to me. In fact, I think no single experience in the study of the calculus had so profound an effect in stimulating my interest. I learned nothing that was wrong from the incomplete form, and much that was right. I received as much of truth as I could accept at that stage. And so I have been glad in my own instruction, here at least, to avoid teaching over the heads of my students. All the better men can get so much of the truth. None would get anything if double limits were brought in.

There are other forms of the theorem, I know, which seem preferable to other mathematicians. That is largely a personal question. What I should like to designate as *Duhamel's Principle* is any theorem, correct within the limits of understanding of the student at that stage, which makes possible the mathematical formulation as definite integrals, of such quantities as moments of inertia, fluid pressures, attractions, etc.

You ask, perhaps, why I wrote the Annals paper of 1903 if I was content with the incomplete form. I did not write it to help the beginner; but it seemed to me that the teacher who is not a specialist in analysis might like to know what the facts are, in order fully to dominate his instruction—and incidentally I did not care to risk having thrown up against me that I did not know the facts myself.

I repeat, then, that my own experience of over thirty years in teaching the calculus leads me to prefer substantially the form of Duhamel's theorem you have used, as being suggestive and inspiring to the beginner.

WILLIAM F. OSGOOD, Harvard University

P.S. I do not follow Ettlinger in his criticism of your example on page 25. Your conclusions are justified by your hypotheses.

III. A NOTE BY ASSOCIATE EDITOR W. B. CARVER

EDITOR'S NOTE. It seems important to the Editor that our readers should see precisely the point on which there is a difference of opinion in this matter. In Professor Osgood's paper of 1903, he showed that in the form of Duhamel's theorem, which is stated on page 22 of Professor Woods' book, the conclusion of the theorem, in certain cases which do not ordinarily arise in practice, would not follow from the hypotheses as stated. This point seems not to be in dispute. Professor Ettlinger has criticized the use of such a form of the theorem, preferring the use of some one of the forms which are rigorously correct but which, in the opinion of many teachers, are beyond the grasp of the students for whom the book is written. This kind of question arises frequently in other connections in mathematics, and it is one of profound scientific and pedagogical significance. Should we state a theorem in a text-book with a hypothesis which, in the light of the most rigorous modern analysis, we know to be incomplete; or should we state it in a complete form which the student will probably not be able to understand? Perhaps we are not driven to either of these alternatives. Is it not possible to give the reader only so much of the truth as he is likely to understand without making any statement which is false even from the most rigorous point of view?

W. B. CARVER, Cornell University

REVIEWS

The Work of the College Entrance Examination Board 1901–1925. Boston, Ginn and Company, 1926. 300 pages. Price, $4.00.

The College Entrance Examination Board celebrated the completion of twenty-five years of work by a dinner in New York on November 6, 1925, and by the publication of a "brief record of its origin, achievements, and aims." The aims are discussed in various connections throughout the book but are summarized on the title page as "the solution of educational problems through the cooperation of all vitally concerned."

The speakers at the dinner included Nicholas Murray Butler, Wilson Farrand, Henry S. Pritchett, and Julius Sachs. These men were able to speak with authority on the history of the formation and development of the Board

and abstracts of their remarks occupy the first part of the book. Two recent articles, *A Brief History of the College Entrance Examination Board*[1], by Wilson Farrand, and *The Art of Examination*[2], by A. Lawrence Lowell, are reprinted in full.

The bulk of the volume consists of a digest of the publications of the Board. In addition to the twenty-five annual reports of the Secretary, the Board has published over one hundred numbered documents containing information on specific topics, and the examination questions have been published in various collected forms by Ginn and Company.

The reports of the Secretary contain a large number of statistical tables showing in detail the work carried on by the Board and presenting many interesting statistical studies of educational problems as reflected in the results of the examinations. In 1901 the Board examined 973 candidates. The number has grown steadily until in 1925 there were 19,775. These figures may be taken as a fair indication of the increase in the general activities of the Board and of its widening influence among both the colleges and the preparatory schools.

The Board's first requirements were based largely upon the report, published in 1899, of the Committee on College Entrance Requirements of the National Education Association. Other requirements in mathematics were adopted in 1903 and were retained for two decades. These were formulated by a committee of the American Mathematical Society which published its report in the Bulletin in November, 1903.

The definition of the requirements adopted in April, 1923, was formulated by a commission of eleven members appointed by the Board. Four of the members were also members of the National Committee on Mathematical Requirements created by the Mathematical Association in 1918. The commission submitted two reports, one covering algebra and trigonometry, the other covering geometry. Both of these reports were widely circulated with a view to securing constructive criticisms from teachers of mathematics, copies being addressed to every name on the subscription lists of the *Mathematics Teacher* and of THE AMERICAN MATHEMATICAL MONTHLY. After revision these reports were adopted and published May 15, 1923, as Documents No. 107 and 108, respectively.

The requirements conform in substance to the recommendations contained in the report of the National Committee published in February, 1923, under the title "The Reorganization of Mathematics in Secondary Education." The report of the National Committee has had a profound influence on the teaching of secondary school mathematics and particularly on the preparation of textbooks in algebra. Unfortunately the report is now out of print and the

[1] *School and Society*, Vol. XXII, No. 568, Nov. 4, 1925.
[2] *Atlantic Monthly*, Jan., 1926.

committee is no longer engaged in its work of spreading propaganda among the vast body of teachers. At the present time the College Board is the most important agency which, by its annual examinations interpreting and emphasizing the new requirements, is actively engaged in carrying on the work started by the Mathematical Association.

The cooperative attitude of the Board, as illustrated by the history of its requirements in mathematics, is apparent throughout the book. Its attitude toward innovations is explicitly stated as follows (p. 204):

"The College Entrance Examination Board endeavers to maintain an open-minded attitude toward any research relating to the principles which underlie its work and toward any changes that are suggested in its organization. It is the policy of the Board to consider no question as closed and no problem as solved if it is regarded as still unsettled by educators in good standing who have given it careful attention."

Naturally there have been some just adverse criticisms; the difficult work of the examiners has not always been perfect; the influence of the examinations may have led to cramming in some schools. But these troubles appear quite insignificant when one appreciates the effects obtained in uniformity of entrance requirements for the colleges of the Board and the high standard of excellence shown by the schools in meeting them.

Considered as a contribution to the literature of importance to every teacher and administrator, the purpose of this book is well described in the preface by the Secretary, Professor Thomas S. Fiske. "Those interested in education will find in these pages two important messages: one is that the solution of educational problems should be sought in the cooperation of all groups, whether of institutions or individuals, whose interests are vitally concerned; the other is that examinations conducted by an agency external to the school and independent of any purely local or provincial authority are not only the best measures of a pupil's attainment but also an instrument of great value for education."

W. R. LONGLEY.

Riemannian Geometry. By L. P. EISENHART. Princeton, University Press, 1926. 262 pages.

This little book makes one more real contribution to the mathematical literature of America. Since so much has been done on this subject by Americans, especially by the author and his followers, it is indeed fitting that Professor Eisenhart should be the first to present the whole subject in English. It is written in a delightful style and should prove an inspiration to many young geometers. The author has promised a later book on Non-Riemannian geometry. Let us hope that it will not be too long in coming and that it will prove as delightful as the present one.

Several books on Riemannian geometry have appeared and in them the element of arc is defined by a positive definite quadratic form but in the present book Riemannian geometry is defined as any geometry with a metric, ds^2, not necessarily a definite form.

The element of arc is then written $ds^2 = e g_{ij} dx^i dx^j$, where e is plus one or minus one and will be so chosen that ds^2 is always positive, or in other words so that ds is always real. The introduction of the multiplier e is a new departure. Einstein defined the element of arc as $ds^2 = g_{ij} dx^i dx^j$, even in the case of an indefinite form, and of course ds is imaginary in certain regions of the space. Eisenhart defines the length of a curve only in regions of the space in which $g_{ij} dx^i dx^j$ does not change sign.

If the quadratic form is definite, a unit vector is defined by the relation

$$\lambda^i \lambda_j = \epsilon_j^i \text{ where } \epsilon_j^i = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j. \end{cases}$$

while if the quadratic form is indefinite, a unit vector is defined by $\lambda^i \lambda_j = e \epsilon_j^i$ where $e = 1$ if $g_{ij} \lambda^i \lambda^j$ is positive and $e = -1$, if $g_{ij} \lambda^i \lambda^j$ is negative.

The book is divided into six chapters. Chapter I, "Tensor Analysis," develops the notions of covariant and contravariant systems or tensors, using the usual notation with slight modifications. Chapter II, "Introduction of a Metric," treats the general topics of orthogonality, parallelism curvature, metric properties of a V_n immersed in a V_m, associate directions, etc. In this chapter the author presents a considerable amount of his own work, especially on spaces which contain fields of parallel vectors. Chapter III, "Orthogonal Ennuples," treats the congruences of curves, Ricci coefficients of rotation, Ricci principal directions, N-tuply orthogonal systems of hypersurfaces and allied subjects. Chapters IV and V, "Geometry of Subspaces," and "Subspaces of a Flat Space" are generalizations of topics usually discussed in ordinary differential geometry. The last chapter is an excellent presentation of groups of motions.

There are good lists of exercises in each chapter which will prove welcome to any teacher who wishes to use the book for a text book, and to students who wish to read the book by themselves. A good bibliography is given at the end.

C. L. E. MOORE

Plane Trigonometry. By LLOYD L. SMAIL. New York, McGraw-Hill Book Co., 1926. xii+203+43 pages. Price complete, $2.25; text only, $1.50, tables only, $1.00.

The demands of the modern college are so varied that it is exceedingly difficult for one to write a text, even in trigonometry, for general use. The

author of this new text seems to have solved the difficulty by dividing his text into two parts. The first part is made up of the fundamentals that would be given in any course in trigonometry and these are presented in a brief, interesting way. The second part "reviews and extends the topics of Part I, but from a more advanced standpoint, and gives alternative proofs and methods of treatment".

The text has two possible beginnings and many possible endings; that is, it is suited to the needs of the instructor giving a brief course or to one giving a more extended course, and also it is suited to the instructor who wishes to begin by defining the ratios first for acute angles or to the one who wishes to begin with the general definitions. The author outlines a secondary arrangement for the needs of the latter group. The instructor who wishes to divide his students into sections in accordance with their ability will find Part I suitable for average students and Part II will furnish sufficient extra material to take care of the needs of the exceptional students.

The text is well written and the topics are presented in a clear and attractive style which has given to an old subject a new appearance. On the inside of the back cover is a pocket in which is inclosed a brief set of four-place tables which will be very convenient for examinations.

A special set of tables has been prepared for the text. These tables are different from those usually found. For instance, in Table IV, the mantissa for each four-place number up to 2000 and for each three-place number up to 999 is given opposite the number. This arrangement, together with tabular differences and tables of proportional parts, makes interpolation easy, but it remains to be seen by trial whether the new arrangement is an improvement over the old or not.

The problems in both parts are good and answers are given to about half of the exercises and so, again, the author has been able to solve satisfactorily a problem in text writing.

On pages 4 and 5 of the Tables, "Table III" is used several times when it is obvious that "Table IV" is meant.

R. P. STEPHENS

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) the titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

American Journal of Mathematics, volume 48, no. 4, October 1926: "Regular maps on an anchor ring" by H. R. Brahana, 225–240; "Determination of the type of tricrunodal quartic by means of its invariants" by L. T. Moore, 241–252; "Subgroups of index p contained in a group of order p^m" by G. A. Miller, 253–256; "Mapping by means of linear systems of curves invariant under Cremona involutions"

by T. L. Bennett, 257–276; "Algebraic and transcendental equations connected with the form of stream lines" by H. Bateman, 277–296.

Annals of Mathematics, volume 28, no. 1, December 1926: "On the problem of coloring maps in four colors, I" by C. N. Reynolds, Jr., 1–15; "Some theorems on deducibility" by C. H. Langford, 16–40; "Note on certain associated systems of linear equalities and inequalities" by L. L. Dines, 41–42; "Some further theorems concerning the formation of chains" by A. Arwin, 43–52; "Note on the extensions of groups to obtain n-th roots" by M. H. Ingraham, 53–58; "Periodic solutions of linear differential equations" by W. B. Fite, 59–64; "On the inversion of the order of integration of a two-fold iterated integral" by H. J. Ettlinger, 65–68; "The summability of a single and multiple Fourier series" by H. W. Bailey, 69–91; "The arithmetic of a general algebra" by O. C. Hazlett, 92–102.

Mathematische Zeitschrift, volume 25, no. 4, December 1926: "The identities of affinely connected manifolds" by T. Y. Thomas, 714–722; "A projective theory of affinely connected manifolds" by T. Y. Thomas, 723–733.

Proceedings of the National Academy of Sciences, U. S. A., volume 12, no. 12, December 1926: "Transformations of manifolds with a boundary" by S. Lefschetz, 737–738; "A possible way to discuss the fundamental principles of relativity" by P. Slavenas, 739–744; "Concerning paths that do not separate a given continuous curve" by R. L. Moore, 745–752; "On simply transitive primitive groups" by W. A. Manning, 753–755; "Quadratic forms which represent all integers" by L. E. Dickson, 756; "Congruences of parallelism of a field of vectors" by L. P. Eisenhart, 757–760; "Concerning certain types of continuous curves" by G. T. Whyburn, 761–766; "Summary of results and proofs concerning Fermat's last theorem" (second note) by H. S. Vandiver, 767–762.

Sitzungsberichte der mathematisch-naturwissenschaftlichen Abteilung der Bayerischen Akademie der Wissenschaften, May-June 1926: "Uber den Grad der Approximation einer analytischen Funktion" by J. L. Walsh, 223–230.

Transactions of the American Mathematical Society, volume 28, no. 4, October 1926: "A boundary value problem for a system of ordinary differential equations of the first order" by G. A. Bliss, 561–584; "On the theory of integral equations with discontinuous kernels" by R. E. Langer, 585–639 "The figuratrix in the calculus of variations" by P. R. Rider, 640–655; "On the existence of fields in Boolean algebras" by B. A. Bernstein, 654–657 "Asymmetric displacement of a vector" by J. M. Thomas, 658–670; "Tensors determined by a hypersurface in Riemann space" by H. Levy, 670–694; "A comparison of the series of Fourier and Birkhoff" by M. H. Stone, 695–762; "Analytic approximations to topological transformations" by P. Franklin and N. Wiener, 762–785.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

THE DENISON MATHEMATICS CLUB, Denison University, Granville, Ohio.

The following programs were given at the meetings of the Club during the year 1925–1926:
October 6, 1925. "My year in Constantinople" by Dr. F. B. Wiley.
October 20. "Inversions" by Miss Anna Peckham.
December 8. Annual Christmas party.
January 5, 1926. "Large numbers" by Dr. F. B. Wiley.
March 2. "Mathematics and music" by George Stiblitz '26.
March 16. "Logarithms of negative numbers" by Dr. F. B. Wiley.
April 6. Reception—formal opening of the new Mathematics Floor.
April 20. "Alternating current circuits" by Professor Coons.

May 4. "Mathematics and physics" by Professor Howe.
May 18. "Functions of angles" by Miss Mattie Tippit.
June 4. Annual mathematics banquet. Speaker, Dr. Henry Blumberg of the Ohio State University.

The officers for the year 1925–1926 were: President, George Stibitz; vice–president, Emily King; secretary, Josephine Deeds; treasurer, Leland Powell.

The officers for the year 1926–1927 are: President, Leland Powell; vice–president, Thelma Weimer; secretary, Ruth Garrett; treasurer, Thomas Parks.

The Denison Mathematics Club is the oldest and one of the largest departmental societies on the campus. Its purpose is to bring to the students of mathematics an opportunity of acquiring a knowledge of those topics rarely discussed in the classroom.

(Report by Miss Ruth Garrett, secretary)

Pi Mu Epsilon, University of Missouri, Columbia, Missouri.

The officers of the Missouri chapter of Pi Mu Epsilon for the year 1925–1926 were: Director, Richard Shewmaker '26; vice–director, Vernon Tiller '27; secretary, Nola Lee Anderson, graduate student; treasurer, George Cramer, '26.

The three highest Phi Beta Kappa students were members of Pi Mu Epsilon. Each year the organization gives a prize to the sophomore or junior who makes the highest grade in a special examination on the elementary calculus.

The program for 1925–1926 was the following:
October 13, 1925. "The place of mathematics" by Dr. W. D. A. Westfall.
October 28. "Rigor in mathematics" by Dr. Louis Ingold.
November 10. "Groups" by Dr. E. F. Allen.
November 24. "Applications of groups" by Dr. G. E. Wahlin.
December 8. "The probability integral" by Mr. George Cramer. "Approximation of the witch by Taylor's series" by Miss Edna Robinson.
February 9, 1926. "Vectors" by Mr. F. O. Duncan.
March 9. "Vectors" continued by Mr. F. O. Duncan.
March 23. "Mathematics in Missouri in 1824" by Miss Kathryn Wyant.
April 13. "Vectors as used in electrical engineering" by Professor Weinbach.
April 27. "Symmetrical functions" by Mr. C. G. Yeager.
May 11. "The history and development of the calculus" by Dr. Betz.

(Report by Miss Katherine Wyant)

The Mathematics Club of Brown University, Providence, Rhode Island.

The program for 1926–27 is announced as follows:
November 11, 1926. "How one once reckoned" by Sadiean Kaull Gladding, '27. "Diophantine equations" by Earl Halliday Bradley, '28.
December 14. "Foucault's pendulum experiment" by Miriam Estelle Ware, '27. "Jerome Cardan" by Alfred William Pett, Jr., '28.
January 11, 1927. "Imaginaries in geometry" by William Caspar Graustein, Associate Professor of Mathematics, Harvard University.
February 22. Mathematical paradoxes" by Isabelle Virginia Rowell, '28. "The slide rule" by Homer Pine Smith, '29.
March 22. "Certain relations between polynomials and integers" by Mark Hoyt Ingraham, Assistant Professor of Mathematics, Brown University.
April 26. "The bicentenary of Newton's death" by Julia Ayer Oldham, '28. "The harmonic analyzer" by Oscar Americus Carnevale, '28.
May. Picnic

Committee on Program: Professor Adams; Nellie Chase Morton, '27; Isabelle Virginia Rowell, '28; Aubrey Henderson Smith, Gr.; Robert Sumner Asbury, '28.

Committee on Arrangements: Mr. Lockwood; Mildred Venetia Mott, '27; Ethel McKechnie, '28; Arthur Rockwell Tebbutt, '27; Alfred William Pett, Jr., '28.

(Report by Professor R. C. Archibald)

The Mathematics Club of the University of North Carolina, Chapel Hill, North Carolina.

The officers for the session 1925—26 were: M. A. Hill, president; V. A. Hoyle, vice-president; W. V. Parker, secretary-treasurer.

The program for the year consisted of the following:

October 26, 1925. "The area under the probability curve" by Dr. J. B. Tinker.

November 24. "Some problems in the modern geometry of the triangle" by Dr. A. W. Hobbs.

January 26, 1926. "Curvature of the plane curve in terms of differentials" by Dr. J. W. Lasley, Jr.

"A proof by means of complex variables of the following theorem. *If the joins of the vertices of two triangles are concurrent, the meets of the corresponding sides are collinear*" by Dr. A. Henderson.

February 23. "A diophantine problem" by E. T. Browne. "Solutions of the pair of equations $x^2+y=7$, $x+y^2=11$," by L. E. Bush.

(Report by Professor Lasley)

The Euclidean Circle of the University of Indiana, Bloomington, Indiana.

The program of the Euclidean Circle of the University of Indiana for the year 1925–1926 was the following:

October 5, 1925. "Development of mathematics in Indiana University" by Professor S. C. Davisson. "Mathematics among Chinese students" by Miss Anna Clark.

October 19. "The life and works of Sir Isaac Newton" by Mr. G. C. Campbell. "Laplace, the second Newton" by Miss Helen Pearson.

November 2. "The history of π" by Mr. L. C. Shugart and "A report on the mathematics section of the State Teachers' Association" by Professor Cora Hennel.

November 16. "Lucretius and mathematics" by Miss Alice Abell. "Probability" by Mr. W. J. Kirkham.

December 14. Social meeting.

January 18, 1926. "The relation between mathematics and poetry" by Miss Edith Bauer.

February 15. "Some modern notions of space and time" by Professor H. T. Davis.

March 1. "The summation of series by the use of Bernoulli's numbers" by Miss Irene Price. "Methods of summing series" by Dr. H. A. Zinszer.

March 22. Social meeting.

April 19. "Diffraction patterns" by Professor M. E. Hufford.

May 3. "Mathematical principles of the theory of wealth" by Mr. V. V. Latshaw.

May 17. Annual picnic.

(Report by Miss Arkys Roberts)

The Newtonian Society, State College of Washington, Pullman, Washington.

The officers for the year 1925–1926 were: Marion Upton, president; Florence Johnson, secretary-treasurer; Dorothy Webster, reporter.

The following programs were given:

October 8, 1925. "Four-dimensional theory" by Professor E. C. Colpitts.

October 22. "Infinitesimals" by Professor C. A. Isaacs.

November 13. "Life insurance policies" by H. H. Irwin.

December 10. "Einstein's theory" by Professor R. O. Hutchinson.

January 14, 1926. "Archimedes' cattle problem" by Uarda Davis. "The first Alexandrian school" by Audrey Graber.

January 27. "The source of stellar energies" by C. D. Calogeris.
February 25. "The physical basis of the musical scale" by Arthur Bond.
March 11. "Non-Euclidean geometry" by Alice Grant.
March 25. "Mathematical puzzles" by Marion Upton and Fern Bolick.

(Report by Professor C. A. Isaacs)

THE MATHEMATICS CLUB OF NORTHWESTERN UNIVERSITY, Evanston, Illinois.

The following programs were given by the Club during the college year 1925–1926:

October 29, 1925. "Modern developments in mathematics" by Professor D. R. Curtiss. Election of officers for the first semester: Joe Potchen, Gr., president; Wm. Bauer, '27, vice-president; Louise Stookey, '26, secretary; Rachel Hawks, '27, treasurer; Professor D. R. Curtiss, faculty adviser.
November 12. "Origin of analytic geometry" by Louise Stookey, '26.
December 3. "π" by Professor F. E. Wood.
December 17. "Mathematics of hydraulics" by Melville Larson, Gr.
January 20, 1926. "Mathematical puzzles" by Rachel Hawks, '27.
February 24. "A demonstration of the slide rule" by Miss Brown of the Keuffel and Esser Company.
March 17. "Orbital motion of the Earth" by Malcolm Bauer, '27. Election of officers for the second semester: Melville Larson, Gr., president; Birney Dysart, '27, vice-president; Janet Bonar, '27, secretary; Phyllis Hayford, '26, treasurer; Professor F. E. Wood, faculty adviser.
April 14. "Magic squares" by Lois Girdner, '28.
April 28. "The three-point problem" by Professor Berger; "Explanation of the abacus" by Mr. Kim.
May 12. "Differentials and integrals of non-integer index" by Joe Potchen, Gr.
May 26. "Harmonic wave analysis" by Birney Dysart, '27.
June 16. Picnic at the Forest Reserve.

(Report by Miss Janet M. Bonar, secretary)

THE MATHEMATICS CLUB OF THE UNIVERSITY OF NEBRASKA, Lincoln, Nebraska.

The membership of the Club for the year 1925–1926 numbered one hundred. The following programs were given:

October 8, 1925. Officers for the first semester were elected and the constitution and by-laws read. "Calculating machines" by Professor W. C. Brenke. Demonstrations were given on a number of different machines.
November 12. "The Eleven Check" by Miss Evelyn Wallwey. Dean Engberg spoke on "Mathematical recreations" giving the formulae for working a number of card tricks, and demonstrating how they worked out.
December 10. Mr. Lester Shoemaker told the club "How to draw a straight line" and illustrated his talk with a Peaucellier cell, which he had constructed, and which he later donated to the Mathematics Club. "A proposed duodecimal number system" by Professor A. L. Candy.
January 14, 1926. Mrs. Florence B. Young discussed the "Problem of Apollonius" giving the constructions and proofs of the nine cases leading to the problem. Mr. Sherer talked on "Mathematics in map making" and illustrated his lecture with lantern slides.
February 11. A social meeting of the club was held and the officers for the second semester were elected.
March 19. "Pascal's theorem" by Miss Irma Wiedeman. Professor M. G. Gaba spoke on the game of "Nim" and after demonstrating how the game is played and won, he explained the theory of the game which depends on using a binary system of numbers.
April 15. Professor Deming of the Department of Chemistry gave an illustrated lecture on "Graphic representation of data."
May 13. "The Importance of the Arabs in the history of science" by Mr. William Cejnar. Professor Runge next spoke on "Some graphic methods of computation."
May 19. A social meeting and picnic was held in Antelope Park.

(Report by Miss F. B. Young, secretary)

The Junior Mathematics Club of the University of Chicago, Chicago, Illinois.

The meetings of the Junior Mathematics Club alternate bimonthly with those of the Graduate Mathematics Club. Meetings for the year 1925–1926 were held as follows:

October 28, 1925. "Pythagoras and modern mathematics" by Professor A. C. Lunn.

November 24. "Maps and map making" by Mr. L. S. Johnston.

January 20, 1926. "Some elementary theorems from the theory of algebraic numbers" by Mr. R. H. Marquis.

February 17. "Some elementary theorems from point set theory" by Mr. Raymond Garver.

March 3. "Numerical solution of algebraic equations" by Mr. J. Williamson.

March 20. Social meeting at the home of Professor H. D. Slaught, in which the faculty and students of the department were guests of the club.

April 21. "The Weierstrass theory of irrational numbers" by Mr. R. H. Bardell.

May 5. "A new expression for the remainder in Taylor's formula" by Mr. L. M. Blumenthal.

(Report by Miss Beenken)

The Mathematics Club of the University of Oregon, Eugene, Oregon.

The officers for the year 1925–1926 were: Eula Benson, president; Evan Lapham, vice-president; Helen White, secretary; Hubert Yearian, treasurer; Helen Shinn, historian.

The officers for the year 1926–1927 are: Helen Shinn, president, Hubert Yearian, vice-president; Edna English, secretary; Edmund Veazie, treasurer; Gladys McCornack, historian.

The programs were the following:

October 25, 1925. Miss Wave Leslie gave an account of her experiences in China where she had taught in the Canton Christian College.

November 19. Mr. Willis, graduate assistant, now deceased, spoke on "Rational plane curves as geometrical loci."

January 26, 1926. Helen Shinn explained the use of the slide rule and gave a short history of it. Lawrence Laveridge discussed Einstein's theory of gravitation.

April 20. Mr. Rojansky, instructor, explained the use of the surge tank.

Other meetings were held and were given over wholly to business and social affairs.

(Report by Miss Edna English, secretary)

Irrational Club of the University of Wyoming, Laramie, Wyoming.

The following meetings of the Club were held:

October 9, 1925. Organization meeting at the home of Mr. O. H. Rechard. Business and social program.

October 29. "Mathematical prodigies" by Miss Lillian Portenier.

November 12. "Non-euclidean geometry" by Robert Burns.

November 24. Continued discussion of non-euclidean geometry, by Clark Biesmier.

December 10. "Determination of π" by Philip Pepoon. "Duplication of the cube" by Mark Taylor.

January 14, 1926. "Elementary mathematics in astronomy" by Dr. Gossard.

February 11. "Mathematical fallacies" by Elizabeth Johnson, Frances Colt, and Stephen Anderson.

February 18. Social meeting in Hoyt Hall.

March 12. "Mathematics for commerce students" by Carl Johnson.

March 26. "Applications of the slide rule" by Ben Bellamy, City Engineer.

April 8. "Archimedes" by George Goemmer. "Newton" by Mark Taylor.

April 28. "The nine point circle" by Marion Preator.

(Report by Miss Frances Colt)

The Tulane Mathematics Club, New Orleans, Lousiana.

The program for the year 1925–1926 was the following:

November 18, 1925. Election of officers for the year: Professor H. E. Buchanan, Tulane University,

president; Professor A. M. Howe, Newcomb College, secretary. Paper on "The mortality table and probability of life" by Mr. E. F. Holtzman, assistant actuary of the Pan-American Life Insurance Company.

March 12, 1926. Joint meeting with the Louisiana-Mississippi section of the Mathematical Association of America, and the New Orleans branch of the American Academy of Science.

May 6, 1926. Paper on "Continuous annuities by means of Gamma Functions" by Mr. William Larkin Duren Jr., Tulane University.

(Report by Professor A. M. Howe)

Pi Mu Epsilon, University of Illinois, Champaign, Illinois.

In addition to two social meetings and the regular bi-monthly business meetings, the following program meetings were held:

November 12, 1926. "Theory of relativity" by Professor R. D. Carmichael.

December 13. "The history of mathematics" by Professor G. A. Miller.

March 11, 1926. "The algebra of logic" by Professor J. B. Shaw.

April 22. "Water particles expanded into steam" by Professor G. A. Goodenough, Department of Engineering.

May 20. "Application of mathematics to astronomy" by Professor R. H. Baker, Department of Astronomy. The following officers were elected for the coming year: Janet Weston, director; J. C. Springer, vice-director; Lydia Hackman, secretary; R. G. Ehman, treasurer; C. F. Robbins, librarian.

(Report by Miss Lydia Hackman, secretary)

The Mathematics Club of Columbia College, New York City.

The Club held two meetings during the academic year 1925–1926, the first in December, the second in January. At the first meeting, Mr. Leonard Sindeband, '27, spoke on the game of "Nim". At the close of this meeting, L. Sindeband '27, A. I. Dumey, '26, and Edward Hymes, '28, were elected members of the Executive Committee. At the second meeting, Mr. A. I. Dumey, '26, spoke on "Inversions". Dr. J. P. Ballantine and Mr. A. E. Meder acted as faculty advisers.

(Report by Professor M. H. Stone)

PROBLEMS AND SOLUTIONS

Edited by B. F. Finkel, Otto Dunkel and H. L. Olson.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

[N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3251. Proposed by J. V. Uspensky, Carleton College.

Show that the system $\phi(x, y) = x + y + \sum A_{i,k} x^i y^k = 1$,

$$x\phi_x' - ay\phi_y' = 0, \text{ where } A_{i,k} > 0,\ a > 0,\ 2 \leqq i + k \leqq n,$$

has one and only one solution in positive numbers.

3252. Proposed by the late Laenas G. Weld.

Find the equation of the plane curve along which a point, moving with a velocity in constant ratio to its ordinate, will pass from (x_1, y_1) to (x_2, y_2) in the least possible time.

3253. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Through the vertices of a given tetrahedron, planes are drawn parallel to the opposite faces. Prove that the faces of the new tetrahedron thus obtained have for their centroids the corresponding vertices of the given tetrahedron.

Note: The new tetrahedron may, by analogy with the case in the plane, be called the "anticomplementary tetrahedron" of the given tetrahedron.

3254. Proposed by C. M. Sparrow, Rouss Physical Laboratory, University of Virginia.

ABC is a triangle with angles α, β, γ. Pairs of lines are drawn through the vertices, symmetrical with respect to the bisectors and making angles $\lambda\alpha, \lambda\beta, \lambda\gamma$ with the sides (λ any real multiple). The two lines adjacent to any side determine a point. Prove that the three points so determined form a triangle which is perspective to the original triangle.

3255. Proposed by R. H. Sciobereti, Berkeley, California.

Find among all the curves of given length and passing through two given points that which has the lowest center of gravity.

3256. Proposed by C. N. Schmall, New York City.

Two spheres, with radii R and r, are inscribed in a right circular cone so that the greater touches the smaller and the base of the cone. If v is the volume of the cone, show that $v = 2\pi R^6/3r(R-r)$.

3257. Proposed by Albert A. Bennett, Lehigh University.

Prove that if V is any real function, of three variables, x, y, z, satisfying the condition that $(\partial^2 V/\partial x^2)+(\partial^2 V/\partial y^2)+(\partial^2 V/\partial z^2)=0$ and expressible as a term-wise differentiable power series, and if the magnitude of the vector $(\partial V/\partial x, \partial V/\partial y, \partial V/\partial z)$ is constant, say equal to unity, then the direction of this vector is also constant. Conversely, if the direction is constant, the magnitude is constant. Note that the restriction to real functions is essential for the first part of the problem but extraneous for the second part.

This is stated without proof as apparently obvious in an old physics book.

SOLUTIONS

3179 [3175; 1926, 159]. Proposed by Otto Dunkel, Washington University.

Is the following statement correct?

The intersections of two curves whose equations are given in polar coördinates (ρ, θ) are obtained by solving the two equations simultaneously for ρ and θ.

Solution by M. S. Knebelman, Princeton University.

The above statement is not correct; for any point (ρ, θ) has also the coördinates $[(-1)^n\rho, n\pi+\theta]$, where n is an integer. Hence to find the points of intersection of two curves whose equations are given in polar coördinates, replace (ρ, θ) by $[(-1)^n\rho, n\pi+\theta]$ in one of them, leaving the other unaltered, and solve simultaneously using all possible integral values of n.

3181 [3177; 1926, 159]. Proposed by Frank Irwin, University of California.

Prove the identity

$$\frac{1}{\sin(y+z)}\left[\sin y+\frac{\sin x \sin z}{\sin(x+y+z)}\right]=\frac{1}{\sin(x+z)}\left[\sin x+\frac{\sin y \sin z}{\sin(x+y+z)}\right].$$

SOLUTION BY A. PELLETIER, Montreal, Canada and BY THE PROPOSER.

If we reduce on the left to a common denominator, the numerator, $\sin(x+y+z)\sin y+\sin x\sin z$,

$$\equiv \tfrac{1}{2}[\cos(x+z)-\cos(x+2y+z)]+\tfrac{1}{2}[\cos(x-z)-\cos(x+z)]$$
$$\equiv \tfrac{1}{2}[\cos(x-z)-\cos(x+2y+z)] \equiv \sin(x+y)\sin(y+z).$$

The left member, therefore, reduces to $\sin(x+y)/\sin(x+y+z)$. From the fact that this is symmetric in x and y the identity follows.

Also solved by W. L. AYRES, THEODORE BENNETT, J. A. BULLARD, A. G. CLARK, B. W. CURRIE, ALICE A. GRANT, MICHAEL GOLDBERG, M. S. KNEBELMAN, HARRY LANGMAN, R. H. SCIOBERETI, and H. S. UHLER.

3182 [3178; 1926, 159]. Proposed by J. L. Riley, Ouachita College, Arkadelphia, Ark.

Find the smallest integral values of x and y which satisfy the equation

$$x^2-67y^2=1,$$

SOLUTION BY H. S. UHLER, Yale University.

The quotients comprised in the first complete period obtained by approximating the binomial surd $\sqrt{67}$ by a continued fraction are 8, 5, 2, 1, 1, 7, 1, 1, 2, 5, 16. The tenth convergent is $48842\div 5967$ and since it falls in an even place the solution of the question is $x=48842$ with $y=5967$.

Remark. The theory of "indeterminate problems of the second degree," the answer to this problem, and a table of solutions of the equation $p^2-Nq^2=1$ for every value of N from 2 to 102, may be found in Peter Barlow's *Theory of Numbers*, London, 1811.

Also solved by S. A. COREY, MICHAEL GOLDBERG, A. L. MCCARTY, A. PELLETIER, and F. L. WILMER.

3187[3175; 1926, 228]. Proposed by J. B. Reynolds, Lehigh University.

Prove that if the perpendiculars from each of three vertices of a tetrahedron upon the opposite faces meet in a point, the fourth will also pass through that point.

I. SOLUTION BY S. B. LITTAUER, Hunter College

Let $ABCD$ be the given tetrahedron with the perpendiculars to the opposite faces, BX, CZ, DY, meeting in O, and let X, Y, Z be their feet in the faces. The plane BOD is perpendicular to the faces BAC and DAC, and it cuts them in the lines BH and DH, each perpendicular to AC. In a similar manner, the plane BOC gives CG and BG, each perpendicular to AD; and then the plane DOC gives CJ and DJ, each perpendicular to AB. The lines CG and DH are altitudes of the face ACD and they meet in X. Hence the plane BOA cuts this face in the third altitude AXF. Since BX is perpendicular to ACD and XF is perpendicular to CD, BF must be perpendicular to CD. Thus the plane BOA is perpendicular to BCD at F. In a similar manner we show that AOD is perpendicular to BCD. Hence their line of intersection, AO, is perpendicular to BCD. This proves the proposition.

II. SOLUTION BY T. C. ESTY, Amherst College

With any origin let $\alpha, \beta, \gamma, \delta$ be the position vectors of the four vertices A, B, C, D, and let ρ be the vector of P, the intersection of the perpendiculars from A, B, C. It will be shown that DP is perpendicular to the face ABC by showing that DP is perpendicular to any two sides of the triangle ABC. Since AP is perpendicular to the face BCD, it must be perpendicular to DC and BD. Hence

(1) $(\rho-\alpha)\cdot(\delta-\gamma)=0$, (2) $(\rho-\alpha)\cdot(\beta-\delta)=0$.

In a similar manner we obtain the equations

(3) $(\rho-\beta)\cdot(\delta-\alpha)=0$, (4) $(\rho-\gamma)\cdot(\alpha-\delta)=0$.

The two pairs of equations (1), (4) and (2), (3) give by addition

$$(\rho-\delta)\cdot(\alpha-\gamma)=0, \qquad (\rho-\delta)\cdot(\beta-\alpha)=0.$$

This completes the proof.

If we add (3) and (4), we obtain $(\beta-\gamma)\cdot(\alpha-\delta)=0$, and this shows that the opposite edges BC and AD are perpendicular. In the same way it may be shown that each of the remaining two pairs of edges are perpendicular.

NOTE BY THE EDITORS: A slight modification of either of the above two proofs shows that: In any trihedral angle the three planes, each of which passes through an edge perpendicular to the opposite face, meet in a straight line through the vertex. This result may be used to establish the theorem of the problem. It is also useful in a proof of 3188 [3176, 1926, 228].

Also solved by NATHAN ALTSHILLER-COURT, MICHAEL GOLDBERG, HARRY LANGMAN, A. PELLETIER, and the PROPOSER.

3188[3176; 1926, 228]. Proposed by J. B. Reynolds, Lehigh University.

Through the vertex, O, of a tetrahedron, $OABC$, are passed three planes perpendicular respectively to OA, OB, and OC. Let BC (extended if necessary) cut the first plane in F, CA cut the second in G, and AB cut the third in H. Prove FGH a straight line.

SOLUTION BY HARRY LANGMAN, Brooklyn, N. Y.

Let O-RST be any trihedral angle, and let the plane RST be perpendicular to OR. Draw $SV\perp RT$ and $TW\perp RS$, cutting SV in Z. Draw RZU, cutting ST in U. Draw OU, OV, OW. The line SV is perpendicular to the plane ORT. Hence, the plane OSV is perpendicular to the plane ORT. Similarly $OTW\perp ORS$. Now $RU\perp ST$ and $RU\perp OR$. We have

$$SU^2+OU^2=SU^2+RU^2+OR^2=RS^2+OR^2=OS^2.$$

$\therefore OU\perp SU$. $\therefore$ Plane $ORU\perp SU$, hence plane $ORU\perp$ plane OST. Hence the planes through the edges of a trihedral angle perpendicular to the opposite faces have a common line of intersection. It follows that the altitudes of a spherical triangle have a common point of intersection—an orthocenter.

The three perpendicular planes form a second trihedral angle O-$A'B'C'$, such that OC' is perpendicular to OAB and OC is perpendicular to $OA'B'$. Hence, the plane of OCC' is perpendicular to both OAB and $OA'B'$. The same thing is true of OBB' and OAA'. These three planes meet in a straight line OZ. The planes OBC and $OB'C'$ meet in a straight line OF which must be perpendicular to both OA and OA'. Hence OF is perpendicular to OZ which lies in the plane OAA'. In the same way it is shown that OG and OH are perpendicular to OZ, and hence these three lines lie in the same plane perpendicular to OZ. This plane cuts the plane of ABC in a straight line which contains the points F, G, H.

Also solved by NATHAN ALTSHILLER-COURT, THEODORE BENNETT, T. C. ESTY, A. PELLETIER, MICHAEL GOLDBERG, and the PROPOSER.

3190[3178; 1926, 229]. Proposed by B. C. Wong, Berkeley, California.

On OP, the radius vector of any point on the cardioid $\rho=2a(1-\cos\theta)$, as diagonal a rectangle $OMPN$ is constructed with the sides OM and ON bisecting the angles between the radius vector and the axis of the cardioid. Find the loci of M and N and give their constructions without the use of the cardioid.

SOLUTION BY MICHAEL GOLDBERG, Washington, D. C.

The equation of M is $\rho'=2a(1-\cos 2\theta')\cos\theta'=4a\sin^2\theta'\cos\theta'$.

The equation of N is $\rho''=-2a[1-\cos(2\theta''-\pi)]\cos\theta''=-4a\cos^3\theta''$.

These equations suggest the following simple geometrical constructions for points of the loci. Lay off $OA=4a$ on one side of the angle θ. Draw AB perpendicular to the other side, BC perpendicular to OA, CD perpendicular to AB, and CE perpendicular to OB. Then $CD=4a\sin^2\theta\cos\theta=\rho'$, and $OE=4a\cos^3\theta=\rho''$. These values of ρ' and ρ'' are associated with the angle θ.

3191[3179, 1926, 229]. Proposed by Einar Hille, Princeton University.

The differential equation $y'=y^{2/3}$ has two independent solutions which pass through the origin, namely $y\equiv 0$ and $y=\frac{1}{27}x^3$. Prove that these solutions can be obtained by the method of successive approximations using as first approximation in the former case $y_0=0$, in the latter case $y_0=x^{a_0}$ where a_0 is any positive constant (x may be assumed to be positive).

Solution by the Proposer.

The successive approximations are determined by the set of equations

$$y_n(x)=\int_0^x [y_{n-1}(t)]^{2/3}dt, \quad n=1,2,\cdots$$

Substituting $y_0(x)=0$, we get $y_1(x)\equiv 0$, hence $y_n(x)\equiv 0$ for every value of n, and $\lim y_n(x)\equiv 0$ which is the first solution.

On the other hand, substituting $y_0(x)=x^{a_0}$, $a_0>0$, we get $y_1(x)=(\frac{2}{3}a_0+1)^{-1}x^{2/3a_0+1}$ and by complete induction $y_n(x)=b_n^{-1}x^{a_n}$, in which $a_n=ra_{n-1}+1$ and $b_n=a_1^{\beta_1}a_2^{\beta_2}a_3^{\beta_3}\cdots a_{n-1}^{\beta_{n-1}}a_n^{\beta_n}$, $(n\geqq 1)$, where $\beta_k=r^{n-k}$ and $r=2/3$.

It remains to prove that $a_n\to 3$ and $b_n\to 27$. We start with a_n. If $a_0=3$, we have $a_1=3$ and consequently $a_n=3$ for every n. Now assume $a_0\neq 3$; then $a_1\neq 3$ and lies between a_0 and 3. Since $a_n-a_{n-1}=\frac{2}{3}(a_{n-1}-a_{n-2})$ for every n, we conclude that the sequence a_n is monotonic, increasing or decreasing according as $a_0<3$ or $a_0>3$. In the former case, 3 is an upper bound of a_n, in the latter case a lower bound. Hence, $\lim a_n$ exists; since $\lim a_n=\frac{2}{3}\lim a_{n-1}+1$, we have $\lim a_n=3$.

We have $b_n=a_n\,b_{n-1}^{2/3}$ or, putting $\log a_n=c_n$, $\log b_n=d_n$, $d_n=\frac{2}{3}d_{n-1}+c_n$. Hence $d_n-d_{n-1}=\frac{2}{3}(d_{n-1}-d_{n-2})+(c_n-c_{n-1})$. Now if $a_n\leqq 3$, $c_n-c_{n-1}\geqq 0$. Thus the d_n sequence is monotonic increasing. But in such a case

$$d_n<\tfrac{2}{3}d_n+c_n<\tfrac{2}{3}d_n+\log 3 \text{ or } d_n<3\log 3.$$

It follows that $\lim d_n$ exists and $\lim d_n=\frac{2}{3}\lim d_{n-1}+\lim c_n$ or $\lim d_n=3\log 3$. Now suppose that $a_0>3$. If the sequence d_n is monotonic increasing the same reasoning applies as in the case $a_0\leqq 3$ with the exception of the estimate of the upper bound where we get $d_n<3\,c_0$ instead of $3\log 3$. The limit of d_n is of course the same. A similar reasoning applies if the sequence d_n is monotonic increasing from a certain point onwards. On the other hand, if the sequence d_n is not increasing from a certain point on, there must exist an n for which $d_n<d_{n-1}$, say $n=k$. But $c_{k+1}-c_k<0$; hence, also $d_{k+1}-d_k<0$ and we find successively that $d_n-d_{n-1}<0$ for $n\geqq k$. Thus the sequence is monotonic decreasing from a certain point on; since $d_n>0$ for every n, $\lim d_n$ exists, its value of course being $3\log 3$.

Hence, $\lim_{n\to\infty} y_n(x)=\lim_{n\to\infty} b_n^{-1}x^{a_n}=\frac{1}{27}x^3$, if $a_0>0$.

3192[3180; 1926, 229]. Proposed by J. Rosenbaum, Milford, Connecticut.

Given the angles of an inscribed polygon and the radius of circumscribed circle to construct the polygon.

Discussion by Nathan Altshiller-Court, University of Oklahoma.

What is wanted is to inscribe in a given circle (O) a polygon having given angles. It is assumed that the order in which these angles are to appear is also given. An elementary solution of this problem depends upon the following:

Lemma. If two broken lines $ABC\cdots$ and $A'B'C'\cdots$ with the same number of sides, N, are inscribed in the same circle so that their sides are respectively parallel, then their closing lines are parallel if N is odd, of equal length if N is even.

The proof is immediate, if we observe that $AA'=BB'=CC'=\cdots$

Suppose now that we want to inscribe a polygon with an odd number of sides, say, a pentagon. Take the line $A'B'$ arbitrarily as a chord of (O), and then draw the chords $B'C'$, $C'D'$, $D'E'$ so that these four chords will make with each other in succession the given angles B', C', D'. Draw the circle (O') concentric with (O) and tangent to $A'E'$. Now inscribe in (O) an angle of the given size A so that one of its sides, AE, is tangent to (O'), and the other, AB, is parallel to $A'B'$. Then draw BC parallel to $B'C'$, CD parallel to $C'D'$. The parallel to $D'E'$ through D will pass through E, since $AE=A'E'$, by construction.

Observe that the size of the angle DEA is determined by the construction; hence, only four of the five angles of the pentagon may be given in the conditions of the problem. This could be seen a priori.

NOTE: If the four given angles have the sum of 360°, the pentagon is degenerate.

If the polygon has an even number of sides, say, six, we may proceed with the construction as above. But the direction of the closing line AF being fixed, according to the lemma, both the angle F and the angle A will be determined by the construction, so that only four of the six angles may be given. The line AF may then be taken arbitrarily, and the problem has an infinite number of solutions, for a given direction of the line AF.

NOTE: The proposed problem may also be stated as follows: In a given circle, to inscribe a polygon so that its sides shall have given directions. Problems of a closely related nature are discussed by E. Catalan in *Théorèmes et problèmes de géométrie élimentaire*, (Dunod, Paris, 1879), pp. 220–225. The problem may be generalized by interpreting *direction* as *point at infinity*, then taking these points at a finite distance, and not necessarily collinear. A very elegant solution, by the method of inversion, of the problem thus stated is to be found in J. Petersen's, *Methods and theories for the solution of problems of geometrical construction*, (Stechert, New York).

The problem may further be generalized by substituting a conic for the given circle. This problem is solved by Poncelet in his famous *Traité des propriétés projectives des figures*, (Paris, 1822), pp. 349–357. Poncelet gives a number of bibliographical references concerning the status of this problem up to his time. Another solution, also based on projective considerations, although different from those of Poncelet was given by M. Chasles in his *Traité des sections coniques* (Gauthier-Villars, Paris, 1865), page 159.[1]

Also solved by MICHAEL GOLDBERG and HARRY LANGMAN.

AN INFORMATION BUREAU FOR APPOINTMENTS

The Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

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[1] See also M. Simon, *Uber die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert* (Teubner Leipzig), pp. 105 etc.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

On the inside of the back cover of this issue of the Monthly is printed a list of "Suggestions to Authors." The editors request that, in the future, authors prepare their manuscripts with those suggestions in mind.

The attention of chairmen of departments is called to the fact that a number of candidates are already registered in the files of the "Information Bureau for Appointments" under the direction of Professor Kuhn. (See page 221.) It is hoped that this bureau will be of real service to departments of mathematics.

The date of the eleventh summer meeting of the Association is Monday and Tuesday, September 5–6, 1927. There will be sessions of the Association in the afternoon of each of these days. The Society will also hold two sessions, namely on Thursday afternoon and Friday morning. The colloquium lectures will be given on Tuesday, Wednesday, Thursday and Saturday mornings and Thursday evening. Wednesday and Friday afternoons are reserved for lake excursions. The joint dinner will be on Wednesday evening. The complete program will be mailed to all members as early as possible in August. Meanwhile we may all mark our calendars for Madison—September fifth to tenth.

Attention should be called again to the basis of cooperation between the Association and the *Annals of Mathematics*. For the past ten years the Association has assisted in subsidizing this journal and in return the Annals has increased the size of its volume by approximately one hundred pages consisting largely of expository papers and has also reduced its subscription price from three dollars to one dollar and fifty cents to members of the Association. Two hundred and eighty members are now taking advantage of this reduced rate privilege. Doubtless many others will be glad to do so when again reminded of the opportunity. Now is an especially opportune time since the Annals is to be still further enlarged through additional subsidy provided by the National Research Council.

The Program and Arrangements committee for the Madison and Nashville meetings are already busily engaged in laying their plans. This is the second occasion when the combined meetings and colloquium have been held at Madison and doubtless there will be a large attendance both on account of the richness of the programs to be offered and because of the many attractions which Madison and its environs provide for those who wish to spend a week in September in such beautiful surroundings. The Nashville meeting, however,

which will be the Annual Meeting of the Society and the Association will be the first to be held so far South. The chief desire of the officers of both organizations in choosing this location was to afford an opportunity for every member of these mathematical bodies in the South and Southwest to attend one of the great national meetings and to share in the inspiration and enthusiasm which such gatherings afford. In order to fully meet these aims it is necessary for large numbers from the North and East to be in attendance but it is especially important that every member from the southern territory should begin to plan now to spend the holiday week in Nashville and to pass the word along to those who are not as yet members but will welcome the invitation to become members and to share in the fellowship which this occasion will afford. The opportunity is unique in another respect in that there will be reduced railroad rates to Nashville from all parts of the country due to the large numbers who will be in attendance upon the meetings of the American Association for the Advancement of Science.

The award of fellowships at the University of Chicago for 1927–28, which has just been announced, contains the names of eight persons who have been appointed in the department of mathematics as follows: ABRAHAM ADRIAN ALBERT, S.B., University of Chicago; MAY MARGARET BEENKEN, Ed.B., University of California, A.M., University of Chicago; BURTON WADSWORTH JONES, A.B., Grinnell College, A.M., Harvard University; EDWARD JAMES MCSHANE, B.E. and S.B., Tulane University; DAVID CLARENCE MORROW, A.B., University of Manitoba, A.M., University of Toronto; GORDON PALL, A.B., University of Manitoba; CHARLES ANDREW RUPP, A.B. and A.M., Harvard University; ROBERT CLARENCE SHOOK, A.B., Miami University.

The National Council of Teachers of Mathematics issued last year its First Year Book on "A General Survey of Progress in the Past Twenty-five Years." It was a book of 200 pages containing nine articles among which was a reprint of E. H. Moore's presidential address on "The Foundations of Mathematics." The Second Year Book has just been published on "Curriculum Problems in Teaching Mathematics" under the editorship of W. D. Reeve of Teacher's College, Columbia University. It contains ten articles covering 297 pages and is divided into three parts dealing with (1) Arithmetic, (2) Junior High School Mathematics, (3) Senior High School Mathematics. Either of these volumes may be purchased of Mr. C. A. Austin, Oak Park High School, Oak Park, Illinois, or of the Bureau of Publications, Teachers College, Columbia University, New York City.

Professor G. D. BIRKHOFF of Harvard University has been elected an honorary member of the Edinburgh Mathematical Society.

Professor J. L. COOLIDGE of Harvard University has been appointed Exchange Professor to France for 1926–27, his term of service to fall in the second half year.

Dr. M. H. STONE of Columbia University has been appointed instructor and member of the Faculty at Harvard University.

Dr. H. W. BRINKMANN of Harvard University has been promoted to an assistant professorship of mathematics.

Associate Professor O. D. KELLOGG of Harvard University has been appointed to a full professorship of mathematics.

Dr. M. S. DEMOS has been appointed instructor at Harvard University.

The following additional courses in mathematics are announced for the summer of 1927.

Brown University. June 20 to July 30. By Professor R. E. LANGER: Theory of integral equations. By Professor M. H. INGRAHAM: Finite groups and the Galois theory of equations.

Cornell University. July 5 to August 13. By Professor J. I. HUTCHINSON: Modern higher algebra. By Professor W. A. HURWITZ: Advanced calculus. By Professor W. B. CARVER: Analytic projective geometry. By Professor D. C. GILLESPIE: Projective geometry. By Professor C. F. CRAIG: Elementary differential equations. Reading and research courses are offered by Professor J. I. HUTCHINSON, Professor VIRGIL SNYDER, Professor F. R. SHARPE, Professor W. A. HURWITZ, Professor W. B. CARVER, Professor D. C. GILLESPIE, and Professor C. F. CRAIG.

University of Missouri. In addition to the usual elementary courses, the following advanced courses are offered: By Professor INGOLD, Modern geometry; Selected topics from geometry. By Professor WAHLIN: Advanced algebra; Selected topics from algebra. By Professor WESTFALL: Calculus; Selected topics from analysis.

University of Wyoming, first term, June 13 to July 20. By Professor O. H. RECHARD: College algebra; Projective geometry; Differential equations. By Professor C. F. BARR: Calculus; Intermediate algebra. Second term, July 21 to August 26. By Professor RECHARD: Calculus; Seminar. By Professor BARR: Solid geometry; Trigonometry; College geometry.

Professor FRANK H. LOUD, for fifty years professor of mathematics and astronomy at Colorado College (emeritus professor since 1907) died March 2, 1927, at St. Petersburg, Florida. He was seventy-five years old.

Sir GEORGE GREENHILL died February 17, 1927.

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I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of _____ Dollars, to be known as the _____ Fund, and to be used for[2] {Endowment—the income only of which may be expended. / Special Projects—for which both principal and income may be expended.}

Witness: _____ Signature _____

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

Two such funds have already been started:
THE CARUS PUBLICATION FUND and
THE ARNOLD BUFFUM CHACE FUND.

Missing Numbers of the Monthly

Cash will be paid for certain single numbers as follows, up to a limited number of copies:

February, March, May or September, 1913; September, 1914; February, March, April or June, 1915; February or Septembe, 1918—fifty cents; September, 1915—seventy-five cents; May, 1915—one dollar (See MONTHLY, March, 1921, p. 152).

Extra copies or volumes of any dates which members wish ot contribute will be used to the best advantage of the Association.

Address all communications to the Secretary, W. D. Cairns, Oberlin, Ohio

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eleventh Summer Meeting of the Association, Madison, Wisconsin, September 5-6, 1927.

Twelfth Annual Meeting, Nashville, Tenn., December, 1927.

The following are dates of Section Meetings of the Association in 1927:

ILLINOIS, Bloomington, Ill., May 13-14.

INDIANA, De Pauw University, April 29-30.

IOWA, University of Iowa, May 6-7.

KANSAS, Topeka, Kan., February 5.

KENTUCKY, Lexington, May 7.

LOUISIANA-MISSISSIPPI, Shreveport, La., March 4-5.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.

MICHIGAN, April.

MINNESOTA, St. Peter, Minn., May 21.

MISSOURI, St. Louis, Mo., November 25-26.

NEBRASKA, Lincoln, May 14.

OHIO, Columbus, Ohio, April 8.

PHILADELPHIA, Philadelphia, Pa., November.

ROCKY MOUNTAIN, Colorado College, April 22-23.

SOUTHEASTERN, Columbia, S. C., April 15-16.

SOUTHERN CALIFORNIA, Los Angeles, Calif., March 12 and November 5.

TEXAS, Not yet determined.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LOUISIANA-MISSISSIPPI BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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